Canonical Models for Representations of Hardy Algebras

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1 Introduction

Our objective in this paper is to describe a model theory for representations of the Hardy algebras, which we defined and studied in [28], that generalizes the model theory of Sz.-Nagy and Foiaş [41] for contraction operators. Our inspiration for this project comes from three sources. The first is the well-known fact that model theory allows one to think of a contraction on Hilbert space as a "quotient" of a "projective" module over $H^{\infty}(\mathbb{T})$. More accurately but still incompletely, one views $H^{\infty}(\mathbb{T})$ as an operator theoretic generalization of the polynomial algebra in one variable $\mathbb{C}[X]$ and one thinks

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of the Hilbert space of the contraction as a module over the algebra it generates, viewing it as a compression of a module over $H^\infty(\mathbb{T})$ that is, essentially, a multiplication representation of $H^\infty(\mathbb{T})$ on a vector-valued H^2-space. Indeed, the $H^\infty(\mathbb{T})$ - $\mathbb{C}[X]$ analogy coupled with model theory has inspired much of operator theory during the last 40 years - and more. We find the "module-over-$H^\infty(\mathbb{T})$" perspective particularly stimulating and we have been especially inspired by the work of Douglas and his collaborators (see, e.g., [12]) and by the work of Arveson [4, 5, 6].

The second source of inspiration for us is the marvelous paper of Pimsner [31] that shows how to build a C^*-algebra, now called a Cuntz-Pimsner algebra, from a "coefficient" C^*-algebra A, say, and a certain type of bimodule E over A, known as a C^*-correspondence. These are denoted $\mathcal{O}(E)$. When $A = \mathbb{C}$ and $E = \mathbb{C}^n$, $\mathcal{O}(E)$ is the famous Cuntz algebra $\mathcal{O}_n$. Sitting inside $\mathcal{O}(E)$ is the norm-closed subalgebra $\mathcal{T}_+(E)$ generated by A and E that we call the *tensor algebra* of E [22]. Indeed, $\mathcal{T}_+(E)$ is a completion of the *algebraic* tensor algebra determined by A and E. For the study of representations of tensor algebras and for other purposes, we were led to consider certain "weak closures" of our correspondences E and to form a "weak completion" of $\mathcal{T}_+(E)$, which we called a *Hardy algebra* and which we denoted $H^\infty(E)$ [28]. When $A = \mathbb{C} = E$, the constructs we are discussing are these: The algebraic tensor algebra is the polynomial algebra $\mathbb{C}[X]$; the tensor algebra $\mathcal{T}_+(E)$ is the disc algebra $A(\mathbb{D})$ viewed as the algebra of continuous functions on the circle that extend to be analytic on the open unit disc; and the Hardy algebra, $H^\infty(E)$, is $H^\infty(\mathbb{T})$. When $A = \mathbb{C}$ and $E = \mathbb{C}^n$, the algebraic tensor algebra is the free algebra in n variables, $\mathbb{C}\langle X_1, X_2, \cdots, X_n\rangle$; $\mathcal{T}_+(E)$ is Popescu's noncommutative disc algebra [34, 35]; and $H^\infty(E)$ is the free semigroup algebra that he defined in [34] and that has been the object of intense study by Davidson and Pitts, and others [10, 9].

And the third source of inspiration comes from the 1947 paper by Hochschild [15], which shows, among other things, that *every* finite dimensional algebra over an algebraically closed field may be expressed as a quotient of a tensor algebra. In fact, in a fashion that is spelled out in [20], if one is interested in studying the representation theory of finite dimensional complex algebras, one may assume that the coefficient algebra is a commutative C^*-algebra. That is, every finite dimensional algebra is Morita equivalent to a quotient of a graph algebra. By this we mean the following: Let $G = (G^0, G^1, r, s)$ be a countable graph with vertex space G^0, edge space G^1 and range and source maps r and s. Then for the C^*-algebra A we take $c_0(G^0)$ and for E we take (a

completion of) the space of finitely supported functions ξ on G^1, which may be view as a bimodule over A via the formula: $a\xi b(\alpha) := a(r(\alpha))\xi(\alpha)b(s(\alpha))$, $a, b \in A$ and $\alpha \in G^1$. If the graph is finite, then the algebraic tensor algebra is the type of algebra to which we just referred. *Every* finite dimensional algebra over $\mathbb{C}$ is naturally Morita equivalent to a quotient of such a tensor algebra. This perspective has dominated much of finite dimensional algebra since Gabriel's penetrating study [13] of algebras of finite representation type. (For a recent survey, see [14].) In general, the Cuntz-Pimsner algebra $\mathcal{O}(E)$ in this setting goes under various names, depending on the structure of the graph, but for the sake of this discussion, $\mathcal{O}(E)$ is simply a Cuntz-Krieger algebra first studied in [8]. The tensor algebra $\mathcal{T}_+(E)$ has been studied by us in [20, 22, 24, 25]. The general theory of Hardy algebras that we developed in [28] was initiated in part to study $H^\infty(E)$ in this setting, and special representations of $H^\infty(E)$, when E comes from a graph, have been studied by Kribs and Power and their co-workers under the name "free semi-groupoid algebras". (See [17].)

The three sources of inspiration combined have become the driving force behind much of our recent work: We want to study tensor algebras and Hardy algebras in a fashion analogous to the theory of contraction operators on Hilbert space with an eye to exploiting the insights from finite dimensional algebra in much the same way that finite dimensional matrix theory and linear algebra inform operator theory. Although our initial focus was on the interactions between operator theory and finite dimensional algebra, we soon realized that the perspective provided significant insights into such things as the theory of (irreversible) dynamical systems [23, 25], the theory of completely positive maps, quantum Markov processes and other aspects of quantum probability[26, 27]. Of course, we are not alone in the appreciation of the impact of Pimsner's insights on these subjects. However, the perspective from non-self-adjoint operator theory and algebras that has been the leitmotif of our work led to useful insights that seem not to be easily accessible from the self adjoint theory.

The theory we present here will be seen to be a direct descendant of the Sz.-Nagy-Foiaş theory spelled out in [41]. However, there is a subtle, yet important, distinction. We present a model theory for *some* representations of our Hardy algebras, not all. We run into the same difficulties that Popescu encountered in [32] and we must limit ourselves to what he called completely non-coisometric representations. We adopt his terminology here. Indeed, our analysis owes a great deal to his work.

In the next section we present background information from [28] and elsewhere that we shall use. In particular, we develop the perspective that the elements in one of our Hardy algebras $H^\infty(E)$ can profitably be studied as functions on the unit ball of the so-called dual of E calculated with respect to a faithful representation of the underlying W^*-algebra. In Section 3, we develop the notion of characteristic operators and functions for completely non-coisometric representations of $H^\infty(E)$ and we show that such representations have canonical models that are (almost) the exact analogue of the models that Sz.-Nagy and Foiaş built for single operators. In Section 4, we prove a model-theoretic analogue of Sarason's original commutant lifting theorem [39] and in Section 5 we identify the relation between invariant subspaces for representations and factorizations of the characteristic functions. Finally, in Section 6, we present an example that shows how our theory functions in a special case related to the classical Sz.-Nagy-Foiaş theory and that helps to clarify the limitations of the "completely non coisometric" hypothesis.

2 Preliminaries

2.1 W^*-Correspondences and Hardy Algebras

We begin by recalling the notion of a W^*-*correspondence*. For the general theory of Hilbert C^*-modules which we use, we will follow [18]. In particular, a Hilbert C^*-module will be a *right* Hilbert C^*-module.

Definition 2.1 *Let M and N be W^*-algebras and let E be a (right) Hilbert C^*-module over N. Then E is called a* (Hilbert) W^*-module *over N in case it is self dual (i.e. every continuous N-module map from E to N is implemented by an element of E). It is called a* W^*-correspondence *from M to N if it is also endowed with a structure of a left M-module via a normal $*$-homomorphism $\varphi : M \to \mathcal{L}(E)$.(Here $\mathcal{L}(E)$ is the algebra of all bounded, adjointable, module maps on E - which is a W^*-algebra when E is a W^*-module [29]). A* W^*-correspondence over M *is simply a W^*-correspondence from M to M.*

An isomorphism of W^*-correspondences E_1, E_2 from M to N is an M,N-linear, surjective, bimodule map that preserves the inner product. We shall write $E_1 \cong E_2$ if such an isomorphism exists.

If E is a W^*-correspondence from M to N and if F is a W^*-correspondence from N to Q, then the balanced tensor product, $E \otimes_N F$ is a W^*-correspondence

from M to Q. It is defined as the self-dual extension [29] of the Hausdorff completion of the algebraic balanced tensor product with the internal inner product given by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \varphi(\langle \xi_1, \xi_2 \rangle_E) \eta_2 \rangle_F$$

for all ξ_1 , ξ_2 in E and η_1 , η_2 in F. The left and right actions of M and Q are defined by

$$\varphi_{E \otimes_N F}(a)(\xi \otimes \eta)b = \varphi_E(a)\xi \otimes \eta b$$

for a in M, b in Q, ξ in E and η in F.

If σ is a normal representation of N on a Hilbert space H and E is a W^*-correspondence from M to N, then H can be viewed as a W^*-correspondence from N to $\mathbb{C}$ and $E \otimes_N H$ is a Hilbert space (with a normal representation of M on it). Of course, $E \otimes_N H$, also denoted $E \otimes_\sigma H$, is nothing but the Hilbert space of the representation of M that is *induced* by σ, $E\text{-}Ind_N^M \sigma$, in the sense of Rieffel's pioneering studies [37, 38]. (See [36, p. 36 ff.] for the general theory.) It is defined by the equation

$$E\text{-}Ind_N^M \sigma(a)(\xi \otimes h) = a\xi \otimes h, \quad \xi \otimes h \in E \otimes_\sigma H, \ a \in M.$$

To lighten the formulas that appear in this paper, we adopt the following notation throughout.

Notation 2.2 *If E is a Hilbert W^*-module over a von Neumann algebra N, if σ a normal representation of N on the Hilbert space H and if $\mathcal{A}$ is any subalgebra of $\mathcal{L}(E)$, then we shall write σ^E for the restriction of $E\text{-}Ind_N^{\mathcal{L}(E)} \sigma$ to $\mathcal{A}$, and for $a \in \mathcal{A}$, we shall often abbreviate $\sigma^E(a)$ as $a \otimes I_H$.*

Note also that, given an operator $R \in \sigma(M)'$, the map that maps $\xi \otimes h$ in $E \otimes_\sigma H$ to $\xi \otimes Rh$ is a bounded linear operator and we write $I_E \otimes R$ for it. In fact, Theorem 6.23 of [37] shows that the commutant of $\sigma^E(\mathcal{L}(E))$ is $\{I_E \otimes R \mid R \in \sigma(M)'\}$.

If $\{E_\alpha\}$ is a family of W^*-correspondences from M to N then one defines the direct sum $\oplus E_\alpha$ as in [29]. It is a W^*-module over N and one defines a left module structure (making it a W^*-correspondence) in a natural way. Combining this observation about direct sums with the notion of tensor products leads us to the Fock space construction: Given a W^*-correspondence E over M, the *full Fock space* over E, $\mathcal{F}(E)$, is defined to be $M \oplus E \oplus E^{\otimes 2} \oplus \cdots$. It

is also a W^*-correspondence over M with the left action φ_∞ (or $\varphi_{E,\infty}$) given by the formula

$$\varphi_\infty(a) = diag(a, \varphi(a), \varphi^{(2)}(a), \cdots),$$

where $\varphi^{(n)}(a)(\xi_1 \otimes \xi_2 \otimes \cdots \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \xi_n$. For $\xi \in E$ we write T_ξ for the creation operator on $\mathcal{F}(E)$: $T_\xi \eta = \xi \otimes \eta,\ \eta \in \mathcal{F}(E)$. Then T_ξ is a continuous, adjointable operator in $\mathcal{L}(\mathcal{F}(E))$. The *norm closed* subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by all the T_ξ's and $\varphi_\infty(A)$ is called *the tensor algebra* of E and is denoted $\mathcal{T}_+(E)$ ([22]). Since $\mathcal{F}(E)$ is a Hilbert W^*-module, $\mathcal{L}(\mathcal{F}(E))$ is a W^*-algebra [29]. Hence the following definition from [28] makes sense.

Definition 2.3 *If E is a W^*-correspondence over a W^*-algebra then closure of $\mathcal{T}_+(E)$ in the w^*-topology on $\mathcal{L}(\mathcal{F}(E))$ is called the* Hardy algebra of E, *and is denoted $H^\infty(E)$.*

The w^*-continuous, completely contractive representations of this algebra are our principal objects of study.

2.2 Representations

Recall that a W^*-correspondence E over a W^*-algebra M carries a natural weak topology, called the σ-topology (see [7]). This the topology defined by the functionals $f(\cdot) = \sum_{n=1}^{\infty} \omega_n(\langle \eta_n, \cdot \rangle)$, where the η_n lie in E , the ω_n lie in the pre-dual of M, M_*, and where $\sum \|\omega_n\| \|\eta_n\| < \infty$.

Definition 2.4 *Let E be a W^*-correspondence over a W^*-algebra N and let H be a Hilbert space.*

(1) A completely contractive covariant representation *of E (or, simply, a* representation *of E) in $B(H)$ is a pair, (T, σ), such that*

(a) σ is a normal representation of N in $B(H)$.

(b) T is a linear, completely contractive map from E to $B(H)$ that is continuous with respect to the σ-topology of [7] on E and the σ-weak topology on $B(H)$, and

(c) T is a bimodule map in the sense that $T(\varphi(a)\xi b) = \sigma(a)T(\xi)\sigma(b)$, $\xi \in E$, and $a, b \in N$.

(2) A completely contractive covariant representation (T, σ) of E in $B(H)$ is called isometric *in case*

$$T(\xi)^*T(\eta) = \sigma(\langle \xi, \eta \rangle),$$

for all ξ, η in E.

The theory developed in [22] applies here to prove that if a representation (T, σ) of E is given, then it determines a contraction $\tilde{T} : E \otimes_\sigma H \to H$ defined by the formula

$$\tilde{T}(\xi \otimes h) = T(\xi)h.$$

Moreover, for every a in N we have

$$\tilde{T}(\varphi(a) \otimes I) = \tilde{T}\sigma^E(\varphi(a)) = \sigma(a)\tilde{T}, \tag{1}$$

i.e., $\tilde{T}$ intertwines σ and $\sigma^E \circ \varphi$. In fact, it is shown in [22] that there is a bijection between representations (T, σ) of E and intertwining operators $\tilde{T}$ of σ and $\sigma^E \circ \varphi$.

It is also shown in [22] that (T, σ) is isometric if and only if $\tilde{T}$ is an isometry.

Remark 2.5 *In addition to $\tilde{T}$ we also require the "generalized higher powers" of $\tilde{T}$. These are maps $\tilde{T}_n : E^{\otimes n} \otimes H \to H$ defined by the equation $\tilde{T}_n(\xi_1 \otimes \ldots \otimes \xi_n \otimes h) = T(\xi_1) \cdots T(\xi_n)h$, $\xi_1 \otimes \ldots \otimes \xi_n \otimes h \in E^{\otimes n} \otimes H$. We call $\tilde{T}_n$ the* n^{th}-power *or the* n^{th}-generalized power *of $\tilde{T}$. An important role in our analysis is played by the following formula which is valid for all positive integers m and n: $\tilde{T}_{n+m} = \tilde{T}_n(I_n \otimes \tilde{T}_m) = \tilde{T}_m(I_m \otimes \tilde{T}_n)$, where I_n is the identity map on $E^{\otimes n}$ [24]. It will also be convenient to write $T_n(\xi) = T(\xi_1) \cdots T(\xi_n)$ for $\xi = \xi_1 \otimes \cdots \otimes \xi_n \in E^{\otimes n}$, so that $\tilde{T}_n(\xi \otimes h) = T_n(\xi)h = T(\xi_1) \cdots T(\xi_n)h$ for $h \in H$.*

The theory developed in [22] shows that there is a bijective correspondence between covariant representations of E and completely contractive representations ρ of $\mathcal{T}_+(E)$ with the property that $\rho \circ \varphi_\infty$ is a normal representation of N. (Given ρ, let $T(\xi) := \rho(T_\xi)$ and let $\sigma(\cdot) = \rho(\varphi_\infty(\cdot))$ then (T, σ) is a representation of E and we write $\rho := T \times \sigma$.) However, *only certain of these extend from $\mathcal{T}_+(E)$ to $H^\infty(E)$*. The full story has yet to be understood, but an initial analysis may be found in [28]. Aspects of the analysis in [28] will play a role in this paper.

The representations of $H^\infty(E)$ that are "induced" by representations of M play a central role in our theory, where they serve as analogues of *pure* isometries. This is made clear in [24] and [28] and will be developed further here.

Definition 2.6 *Let E be a correspondence over a W^*-algebra M and let σ_0 be a (normal) representation of M on a Hilbert space H. The representation of $H^\infty(E)$ on $\mathcal{F}(E) \otimes_{\sigma_0} H$* induced *by σ_0 is defined to be the restriction to $H^\infty(E)$ of $\sigma_0^{\mathcal{F}(E)}$.*

Observe that the covariant representation (T, σ) determined by $\sigma_0^{\mathcal{F}(E)}$ is given by the formulae

$$\sigma = \sigma_0^{\mathcal{F}(E)} \circ \varphi_\infty = \varphi_\infty \otimes I_H \tag{2}$$

and

$$T(\xi) = \sigma_0^{\mathcal{F}(E)}(T_\xi) = T_\xi \otimes I_H, \tag{3}$$

$\xi \in E$. We also say that (T, σ) is *induced by σ_0*.

Remark 2.7 *It follows from Theorem 6.23 of [37] that $\sigma^{\mathcal{F}(E)}$ is a faithful representation of $H^\infty(E)$ if σ is a faithful representation of M. Most of the time, we will be dealing with faithful representations of M, and when non-faithful representations may arise we will go to great lengths to supplement them to yield faithful representations. (See Definition 3.15 and the discussion related to it.)*

2.3 Duals and Commutants

In order to identify the commutant of an induced representation, we introduced concept of "duality" for correspondences in [28]. Since it plays an important role in the present investigation, we outline its salient features. Given a W^*-correspondence E over the W^*-algebra M and given a *faithful* normal representation σ of M on a Hilbert space H, we set

$$E^\sigma = \{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = (\varphi(a) \otimes I_H)\eta, \ a \in M\}.$$

Then E^σ is a bimodule over $\sigma(M)'$ where the right action is defined by $\eta S = \eta \circ S$ and the left action by $S \cdot \eta = (I_E \otimes S) \circ \eta$, for $\eta \in E^\sigma$ and $S \in \sigma(M)'$. In fact, E^σ is a W^*-correspondence over $\sigma(M)'$, where the inner product is

defined by the formula $\langle \eta_1, \eta_2 \rangle = \eta_1^* \eta_2$. This correspondence is called the *σ-dual (correspondence) of E*. Write ι for the identity representation of $\sigma(M)'$ on H. Then we may form the W^*-correspondence $(E^\sigma)^\iota$ over $\sigma(M)'' = \sigma(M)$. Since σ is faithful we can view this as a correspondence over M. As we shall outline, $(E^\sigma)^\iota$ is naturally isomorphic to E in a way that sets up an isomorphism between the commutant of the representation of $H^\infty(E)$ induced by σ and the image of $H^\infty(E^\sigma)$ under the representation induced by ι. The latter acts on $\mathcal{F}(E^\sigma) \otimes_\iota H$.

For a given $\xi \in E$ we define the operator $L_\xi : H \to E \otimes_\sigma H$ by the equation $L_\xi h = \xi \otimes h$. It is evident that L_ξ is a bounded operator and that its adjoint is given by the formula $L_\xi^*(\zeta \otimes h) = \sigma(\langle \xi, \zeta \rangle)h$ for $\zeta \in E$ and $h \in H$.

Proposition 2.8

(i) *[28, Theorem 3.6]For every $\xi \in E$ let $\hat{\xi} : H \to E^\sigma \otimes_\iota H$ be defined by adjoint equation,*

$$\hat{\xi}^*(\eta \otimes h) = L_\xi^*(\eta(h)) \in H,$$

$\eta \otimes h \in E^\sigma \otimes H$. Then $\hat{\xi} \in (E^\sigma)^\iota$ and the map $\xi \mapsto \hat{\xi}$ is an isomorphism of W^*-correspondences (that is, it is a bimodule map and an isometry).

(ii) *[28, Lemma 3.7]For two W^*-correspondences E_1 and E_2 over M,*

$$(E_1 \oplus E_2)^\sigma \cong E_1^\sigma \oplus E_2^\sigma$$

and

$$(E_1 \otimes_M E_2)^\sigma \cong E_2^\sigma \otimes_{\sigma(M)'} E_1^\sigma.$$

The second isomorphism is given by the map that sends $\eta_2 \otimes \eta_1 \in E_2^\sigma \otimes_{\sigma(M)'} E_1^\sigma$ to $(I_{E_1} \otimes \eta_2)\eta_1 \in (E_1 \otimes_M E_2)^\sigma$.

Concerning part (i) of Proposition 2.8, it should be noted that since $\eta \in E^\sigma$, η is an operator from H to $E \otimes_\sigma H$. Thus $\eta(h) \in E \otimes_\sigma H$ for all $h \in H$ and $L_\xi^*(\eta(h))$ makes good sense as an element of H.

With the notation we have established, we also have

Proposition 2.9 *In the notation of Proposition 2.8, the formula*

$$U_k(\xi \otimes h) = \hat{\xi}(h),$$

$\xi \in E^{\otimes k}$, $h \in H$, defines a Hilbert space isomorphism U_k from $E^{\otimes k} \otimes_\sigma H$ onto $(E^\sigma)^{\otimes k} \otimes_\iota H$. The inverse is given by the formula $U^*(\eta \otimes h) = \eta(h)$, $\eta \otimes h \in (E^\sigma)^{\otimes k} \otimes_\iota H$. The direct sum of the U_k, $U := \sum_{k \geq 0}^{\oplus} U_k$, is a Hilbert space isomorphism from $\mathcal{F}(E) \otimes_\sigma H$ onto $\mathcal{F}(E^\sigma) \otimes_\iota H$.

The following result, [28, Theorem 3.9], identifies the commutant of an induced representation in the fashion promised. The theorem is an analogue of the assertion that the commutant of the unilateral shift is the weakly closed algebra generated by the unilateral shift. In Theorem 4.1 it will be generalized to the "model-theoretic" version of the commutant lifting theorem proved by Sarason [39].

Theorem 2.10 *Let E be a correspondence over the W^*-algebra M and let $\sigma : M \to B(H)$ be a faithful normal representation of M on the Hilbert space H. Write $\sigma^{\mathcal{F}(E)}$ for the representation of $H^\infty(E)$ on $\mathcal{F}(E) \otimes_\sigma H$ induced by σ, write $\iota^{\mathcal{F}(E^\sigma)}$ for the representation of $H^\infty(E^\sigma)$ on $\mathcal{F}(E^\sigma) \otimes_\iota H$ induced by the identity representation ι of $\sigma(M)'$ on H and write $U : \mathcal{F}(E) \otimes_\sigma H \to \mathcal{F}(E^\sigma) \otimes_\iota H$ for the Hilbert space isomorphism described in Proposition 2.9. Then the commutant of $\sigma^{\mathcal{F}(E)}(H^\infty(E))$ is $U^* \iota^{\mathcal{F}(E^\sigma)}(H^\infty(E^\sigma))U$.*

Extended Remark and Notation 2.11 *One of the principal achievements of [28] was the representation of elements of $H^\infty(E)$ as functions on the open unit ball of E. This representation plays a role here, but* with a twist. *To understand what we need in more detail, assume that σ is a faithful representation of M in $B(H)$ and let η be an operator in the open unit ball of E^σ, then $\eta^* : E \otimes_\sigma H \to H$ intertwines $\varphi(a) \otimes I_H$ and $\sigma(a)$ for every $a \in M$. Thus, there is a representation (T, σ) of E such that $\eta^* = \tilde{T}$ [22, Lemma 2.16]. Since $\|\tilde{T}\| < 1$ the representation $T \times \sigma$ of $\mathcal{T}_+(E)$ on H can be extended to a σ-weakly continuous representation, also written $T \times \sigma$, of $H^\infty(E)$ (see [28, Corollary 2.14]). So, given $X \in H^\infty(E)$, we define*

$$X(\eta) = (T \times \sigma)(X) \in B(H).$$

That is, each $X \in H^\infty(E)$ gives a $B(H)$-function defined on the open unit ball of E^σ. The properties of this functional representation of $H^\infty(E)$ are explored in [28] . We point out, however, that in general the functional representation of $H^\infty(E)$ is not faithful. That is, $X(\eta)$ can vanish for all η in the open unit ball of E^σ without $X = 0$ [28]. Nevertheless, the function theoretic point of view proves very effective for studying and unifying a wide variety of problems

in operator theory. In particular, in [28], we proved a general version of the Nevanlinna-Pick interpolation theorem, which contains an enormous number of operator theoretic variants of the classical result as a special cases.

In this paper, we shall use the identification of E with $(E^\sigma)^\iota$ through the map $\xi \mapsto \hat{\xi}$ in part (i) of Proposition 2.8 to view elements of $H^\infty(E^\sigma)$ as functions on the open unit ball of E. More importantly, we shall use the spatial identification of the commutant of $\sigma^{\mathcal{F}(E)}(H^\infty(E))$ with $\iota^{\mathcal{F}(E^\sigma)}(H^\infty(E^\sigma))$, given in terms of U and described in Theorem 2.10, to view elements in $(\sigma^{\mathcal{F}(E)}(H^\infty(E)))'$ as functions on the open unit ball of E.

Thus, we adopt the following notation: If $\Psi \in (\sigma^{\mathcal{F}(E)}(H^\infty(E)))'$, then $\hat{\Psi}$ will denote the element in $H^\infty(E^\sigma)$ defined by the formula

$$\hat{\Psi} := (\iota^{\mathcal{F}(E^\sigma)})^{-1}(U\Psi U^*), \tag{4}$$

where $U : \mathcal{F}(E)\otimes_\sigma H \to \mathcal{F}(E^\sigma)\otimes_\iota H$ is the Hilbert space isomorphism defined in Proposition 2.9. Note that $\iota^{\mathcal{F}(E^\sigma)}$ is faithful since ι is (Remark 2.7). We shall also write equation (4) as

$$\hat{\Psi} \otimes I_H = U\Psi U^*. \tag{5}$$

We shall then want to evaluate $\hat{\Psi}$ on the open unit ball of E. On the other hand, given an element $\Xi \in H^\infty(E^\sigma)$, we shall write $\check{\Xi}$ for the operator in the commutant of $\sigma^{\mathcal{F}(E)}(H^\infty(E))$ given by the formula

$$\check{\Xi} := U^*\iota^{\mathcal{F}(E^\sigma)}(\Xi)U = U^*(\Xi \otimes I_H)U. \tag{6}$$

Thus, evidently, we have $(\hat{\Psi})\check{} = \Psi$ and $(\check{\Xi})\hat{} = \Xi$.

This notation is, of course, suggestive of the idea that the Hilbert space isomorphism U in Proposition 2.9 should be viewed as some sort of generalized Fourier transform. The analogy turns out to be more than one built from notation. Accordingly, we shall call $U : \mathcal{F}(E) \otimes_\sigma H \to \mathcal{F}(E^\sigma) \otimes_\iota H$ the Fourier transform determined by *σ. Also, given $\Psi \in (\sigma^{\mathcal{F}(E)}(H^\infty(E)))'$, we shall $\hat{\Psi}$ the* Fourier transform *of Ψ, if $\Xi \in H^\infty(E^\sigma)$, then $\check{\Xi}$ will be called the* inverse Fourier transform *of Ξ.*

3 Characteristic Operators and Characteristic Functions of Representations

In the model theory for a single contraction operator on Hilbert space, the role of the characteristic operator function is to "locate" the Hilbert space

of the operator in the Hilbert space of its minimal isometric dilation. In [22] we successfully constructed isometric dilations of representations of $H^\infty(E)$. (Actually, in [22] we worked with C^*-correspondences over C^*-algebras. Adjustments necessary to handle representations of $H^\infty(E)$, when E is a W^*-correspondence, were made in [28].) We therefore begin by briefly recapping aspects of the theory we shall use.

3.1 Isometric Dilations

Let E be a W^*-correspondence over a W^*-algebra M and let (T,σ) be a completely contractive covariant representation of E on a Hilbert space H. Then (T,σ) has a "minimal isometric dilation", (V,ρ), defined as follows. Recall that the map $\tilde{T}: E\otimes_\sigma H \to H$ defined by the equation $\tilde{T}(\xi\otimes h) = T(\xi)h$ is a contraction that satisfies the equation $\tilde{T}(\varphi(a)\otimes I_H) = \sigma(a)\tilde{T}$. We set $\Delta := (I-\tilde{T}^*\tilde{T})^{1/2}$ (in $B(E\otimes_\sigma H)$), $\Delta_* := (I-\tilde{T}\tilde{T}^*)^{1/2}$ (in $B(H)$), $\mathcal{D} := \overline{\Delta(E\otimes_\sigma H)}$ and $\mathcal{D}_* := \overline{\Delta_*(H)}$. Observe that on account of the intertwining equation $\tilde{T}(\varphi(a)\otimes I_H) = \sigma(a)\tilde{T}$, $\mathcal{D}_*$ reduces σ, while $\mathcal{D}$ reduces $\varphi(\cdot)\otimes I_H = \sigma^E\circ\varphi(\cdot)$. Also we write $D(\xi) := \Delta\circ L_\xi : H \to E\otimes_\sigma H$, $\xi\in E$, where, recall, $L_\xi : H\to E\otimes_\sigma H$ is the map $L_\xi h = \xi\otimes h$, $h\in H$, $\xi\in E$. Note, too, that $T(\xi) = \tilde{T}\circ L_\xi$.

The representation space K of (V,ρ) is

$$\begin{aligned} K &= H\oplus\mathcal{D}\oplus(E\otimes_{\sigma_1}\mathcal{D})\oplus(E^{\otimes 2}\otimes_{\sigma_1}\mathcal{D})\oplus ... \\ &= H\oplus\mathcal{F}(E)\otimes_{\sigma_1}\mathcal{D} \end{aligned}$$

where σ_1 is defined to be the restriction to $\mathcal{D}$ of $\varphi(\cdot)\otimes I_H$. The representation ρ, in the isometric dilation (V,ρ) for (T,σ), is defined to be $\rho = \sigma\oplus\sigma_1^{\mathcal{F}(E)}\circ\varphi_\infty$. That is, $\rho = diag(\sigma,\sigma_1,\sigma_2,\ldots)$ where $\sigma_{k+1}(\cdot) = \sigma_1^{E^{\otimes k}}\circ\varphi_k(\cdot) = \varphi_k(\cdot)\otimes I_{\mathcal{D}}$ acting on $E^{\otimes k}\otimes_{\sigma_1}\mathcal{D}$. The map $V: E\to B(K)$ is defined in terms of the matrix

$$V(\xi) = \begin{pmatrix} T(\xi) & 0 & 0 & \cdots \\ D(\xi) & 0 & 0 & \cdots \\ 0 & L_\xi & 0 & \\ 0 & 0 & L_\xi & \\ & & & \ddots \\ & & & \end{pmatrix}, \tag{7}$$

where we abuse notation slightly and write L_ξ also for the map from $E^{\otimes m}\otimes_{\sigma_1}$

$\mathcal{D}$ to $E^{\otimes(m+1)} \otimes_{\sigma_1} \mathcal{D}$ defined by the equation $L_\xi(\eta \otimes h) = (\xi \otimes \eta) \otimes h$, $\eta \otimes h \in E^{\otimes m} \otimes_{\sigma_1} \mathcal{D}$.

Definition 3.1 *Let E be a W^*-correspondence over the W^*-algebra M and let (T, σ) be a completely contractive covariant representation of E on the Hilbert space H. Then the isometric covariant representation (V, ρ) just constructed is called the* minimal isometric dilation of (T, σ).

The representation (V, ρ) *is* minimal in the sense that the smallest subspace of K that contains H and reduces the set of operators $\{V(\xi) \mid \xi \in E\} \cup \rho(M)$ is all of K. Thus the terminology is justified. We note also that (V, ρ) is unique up to unitary equivalence [22, Proposition 3.2].

If we let $\tilde{V} : E \otimes_\rho K \to K$ be the map that sends $\xi \otimes k$ to $V(\xi)k$, then $\tilde{V}$ be written as the infinite matrix

$$\tilde{V} = \begin{pmatrix} \tilde{T} & 0 & 0 & \cdots \\ \Delta & 0 & 0 & \\ 0 & I & 0 & \\ 0 & 0 & I & \\ & & & \ddots \\ & & & \end{pmatrix}, \tag{8}$$

where the identity operators are interpreted as the maps that identify $E \otimes_{\sigma_{n+1}} (E^{\otimes n} \otimes_{\sigma_1} \mathcal{D})$ with $E^{\otimes(n+1)} \otimes_{\sigma_1} \mathcal{D}$. It is then an easy calculation to see $\tilde{V}^* \tilde{V} = I$ on K, so that $\tilde{V}$ is an isometry (which confirms our assertion that (V, ρ) is an isometric dilation of (T, σ)), and that

$$\tilde{V}\tilde{V}^* = \begin{pmatrix} \tilde{T}\tilde{T}^* & \tilde{T}\Delta^* & 0 & \cdots \\ \Delta\tilde{T}^* & \Delta^2 & 0 & \\ 0 & 0 & I & \\ & & & \ddots \\ & & & \end{pmatrix}, \tag{9}$$

a calculation that we shall use in a moment. Let $\tilde{T}_n : E^{\otimes n} \otimes H \to H$ be the n^{th}-generalized power of $\tilde{T}$ (Remark 2.5) and similarly let $\tilde{V}_n$, mapping $E^{\otimes n} \otimes K$ to K be the n^{th}-generalized power of $\tilde{V}$. Then, of course, each $\tilde{T}_n$ is a contraction, while each $\tilde{V}_n$ is an isometry. Also, as we mentioned in Remark 2.5, $\tilde{V}_{n+m} = \tilde{V}_n(I_n \otimes \tilde{V}_m) = \tilde{V}_m(I_m \otimes \tilde{V}_n)$, where I_n is the identity map

on $E^{\otimes n}$. The importance of the $\tilde{V}_n$ for our purposes is that they implement endomorphisms of the *commutant* of $\rho(M)$. Indeed, if we set

$$L(x) = \tilde{V}(I_E \otimes x)\tilde{V}^*,$$

$x \in \rho(M)'$, then L is an endomorphism of $\rho(M)'$ and

$$L^n(x) = \tilde{V}_n(I_E \otimes x)\tilde{V}_n^*,$$

for all $n \geq 0$ and $x \in \rho(M)'$[24, Lemma 2.3]. It follows easily that for a subspace $\mathcal{M}$ of K that is invariant under $\rho(M)$, the range of $L^n(P_{\mathcal{M}})$ is the span

$$\overline{span}\{V(\xi_1)\cdots V(\xi_n)h : h \in \mathcal{M}, \xi_1, ..., \xi_n \in E\}.$$

Definition 3.2 *A subspace $\mathcal{M}$ of K that is invariant for $\rho(M)$ is called a* wandering subspace*, and the projection $P_{\mathcal{M}}$ of K onto $\mathcal{M}$ is called a* wandering projection*, if for every $n \neq m$, $L^n(P_{\mathcal{M}})$ and $L^m(P_{\mathcal{M}})$ are orthogonal projections. For such a subspace we shall write $L_\infty(\mathcal{M})$ for the range of $\sum_{n\geq 0}^{\oplus} L^n(P_{\mathcal{M}})$.*

Note that, whenever $\mathcal{M} \subseteq K$ is a wandering subspace, the map $W_{\mathcal{M}} : \mathcal{F}(E) \otimes_\rho \mathcal{M} \to L_\infty(\mathcal{M})$ defined by sending $\xi_1 \otimes \cdots \otimes \xi_n \otimes k \in E^{\otimes n} \otimes_\rho \mathcal{M}$ to $V(\xi_1)\cdots V(\xi_n)k \in L_\infty(\mathcal{M})$ is a Hilbert space isometry. Note, too, that for $a \in M$ and $\xi \in E$, we have

$$W_{\mathcal{M}}(\varphi_\infty(a) \otimes I_{\mathcal{M}}) = \rho(a)W_{\mathcal{M}} \tag{10}$$

and

$$W_{\mathcal{M}}(T_\xi \otimes I_{\mathcal{M}}) = V(\xi)W_{\mathcal{M}}. \tag{11}$$

We also write P_n for $\tilde{V}_n\tilde{V}_n^*$, so that $P_n = L^n(I)$. Of course P_1 is given by the matrix (9). Then $\{P_n\}_{n=1}^\infty$ is a decreasing sequence of projections in $\rho(M)'$. We set $Q_n = P_n - P_{n+1}$ and $Q_0 = I - P_1$, so that $\sum_{k=0}^\infty Q_k = I - P_\infty$, where $P_\infty = \wedge P_n$. By [24, Corrolary 2.4], Q_0 is a wandering projection, $Q_k = L^k(Q_0)$ and $Q_\infty := \sum_{k=0}^\infty L^k(Q_0) = \sum_{k=0}^\infty Q_k = I - P_\infty$.

Lemma 3.3 *With the notation just established, we have for every $\xi \in E$ and $m \geq 0$,*

$$V(\xi)Q_m = Q_{m+1}V(\xi)$$

and

$$V(\xi)Q_\infty = Q_\infty V(\xi).$$

Proof. For $k \in K$ we have $V(\xi)Q_m k = \tilde{V}(\xi \otimes Q_m k) = \tilde{V}(I \otimes Q_m)(\xi \otimes k) = \tilde{V}(I \otimes Q_m)\tilde{V}^*\tilde{V}(\xi \otimes k) = Q_{m+1}V(\xi)k$. $\square$

If we let ρ_0 be the restriction of ρ to the range of Q_0, then it follows from [24, Theorem 2.9] that (V, ρ) may be written as the direct sum

$$(V, \rho) = (V_{ind}, \rho_{ind}) \oplus (V_\infty, \rho_\infty)$$

where (V_{ind}, ρ_{ind}) is (unitarily equivalent to) the representation of E that is induced by ρ_0, while (V_∞, ρ_∞) is the restriction to $P_\infty(K)$ and is fully coisometric in the sense of [22, 24, 28], meaning that $\tilde{V}_\infty$ is a coisometry. Thus, $\tilde{V}_\infty$ is a unitary operator on $P_\infty(K)$.

3.2 C.N.C. and C.$_0$ Representations

Our goal is to describe how H sits in the dilation space K. The analysis we present follows Sz.-Nagy and Foiaş, as one might imagine. However, there are some important refinements that are due to Popescu [32] and we need to extend these to our situation. As a first step, we have the following observation, which may be "dug out of" [28] (see Lemma 7.8, in particular.) However, since we need a bit more than is explicit there, we present a proof.

Lemma 3.4 *Write K_0 for the range, $Q_0(K)$, of the projection Q_0. Then*

(i) $K_0 = \overline{Q_0(H)} = \overline{\{\Delta_*^2 h \oplus (-\Delta \tilde{T}^* h) : h \in H\}} \subseteq H \oplus \mathcal{D}$.

(ii) The map u that sends $\Delta_*^2 h \oplus (-\Delta \tilde{T}^* h)$ to $\Delta_* h$ is an isometry from K_0 onto $\mathcal{D}_*$.

(iii) The equation $\rho(a)u = \sigma(a)u = u\rho(a)$ holds for all $a \in M$.

Proof. ¿From the minimality of K it follows that $I_K = \vee_{n=0}^{\infty} L^n(P_H) = \overline{P_H \vee P_1}$. Since Q_0 and P_1 are orthogonal, by definition, we have $Q_0(K) = \overline{Q_0(H)}$. The other equality follows when we write Q_0 matricially as

$$Q_0 = I - \tilde{V}\tilde{V}^* = \begin{pmatrix} I_H - \tilde{T}\tilde{T}^* & -\tilde{T}\Delta & 0 & \dots \\ -\Delta\tilde{T}^* & I - \Delta^2 & 0 & \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \\ & & & \end{pmatrix},$$

as we may, by equation (9). This proves (i). For (ii) we compute:

$$\begin{aligned}\langle \Delta_*^2 h \oplus (-\Delta \tilde{T}^* h), \Delta_*^2 h \oplus (-\Delta \tilde{T}^* h)\rangle &= \langle \Delta_*^4 h, h\rangle + \langle \tilde{T}\Delta^2 \tilde{T}^* h, h\rangle \\ &= \langle \Delta_*^2(\Delta_*^2 + \tilde{T}\tilde{T}^*)h, h\rangle = \langle \Delta_*^2 h, h\rangle,\end{aligned}$$

which proves the assertion. The proof of part (iii) is immediate from the following computation, which is valid for all $a \in M$ and $h \in H$:

$$\begin{aligned}\rho(a)(\Delta_*^2 h \oplus (-\Delta \tilde{T}^* h)) &= \sigma(a)\Delta_*^2 h \oplus (\varphi(a) \otimes I_H)(-\Delta \tilde{T} * h) \\ &= \Delta_*^2 \sigma(a) h \oplus (-\Delta(\varphi(a) \otimes I_H)\tilde{T}^* h) = \Delta_*^2 \sigma(a) h \oplus (-\Delta \tilde{T}^* \sigma(a) h).\end{aligned}$$

□

The following terminology is adopted from [32, 33], which, in turn, derives from the work of Sz.-Nagy and Foiaş (see [41]).

Definition 3.5

(i) A covariant representation (T, σ) will be called a C$._0$-representation *if $P_\infty = 0$ (equivalently, if $K = L_\infty(K_0)$).*

(ii) A covariant representation (T, σ) will be called completely non coisometric *(abbreviated c.n.c.) in case $K = L_\infty(K_0) \vee L_\infty(\mathcal{D})$.*

Remark 3.6 *It is shown in Remark 7.2 of [28] that given a covariant representation (T, σ) of E on a Hilbert space H, then H may be written as $H = H_1 \oplus H_2$ so that if T and σ are written as matrices relative to this decomposition, then*

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix},$$

i.e., σ is reduced by H_1 and H_2, and

$$T(\cdot) = \begin{pmatrix} T_1(\cdot) & 0 \\ X(\cdot) & T_2(\cdot) \end{pmatrix},$$

where (T_1, σ_1) is a covariant representation that is c.n.c. and where (T_2, σ_2) is a covariant representation with the property that all the generalized powers of $\tilde{T}_2$ are coisometries. Further, H_2 may be described as $\{h \in H \mid \left\|\tilde{T}_n^* h\right\| = \|h\|$ for all *$n\}$, i.e., H_2 is the largest space on which all the generalized powers $\tilde{T}_n^*$ act isometrically. Thus (T, σ) is c.n.c. if and only if there is no non-zero vector h such that $\left\|\tilde{T}_n^* h\right\| = \|h\|$ for all n.*

For our purpose here, the significance of the concept "c.n.c." is the condition in the second of the following two lemmas. The first is Proposition 7.15 of [28], while the second is Lemma 7.10 of [28].

Lemma 3.7 *Let (T, σ) be a covariant representation of a W^*-correspondence on a Hilbert space H and let (V, ρ) be its minimal isometric dilation acting on K. Then the following conditions are equivalent.*

(i) (T, σ) *is of class* $C_{\cdot 0}$, *i.e.* $P_\infty = 0$.

(ii) $\wedge \tilde{V}_k \tilde{V}_k^* = 0$, *which happens if and only if* $\|\tilde{V}_k^* k\| \to 0$ *for all* $k \in K$.

(iii) $\tilde{T}_k \tilde{T}_k^* \to 0$ *in the weak operator topology on* $B(H)$, *which happens if and only if* $\|\tilde{T}_k^* h\| \to 0$ *for all* $h \in H$.

(iv) (V, ρ) *is an induced representation.*

So, in particular, if $\|\tilde{T}\| < 1$ *then* (T, σ) *is a* $C_{\cdot 0}$*-representation.*

Lemma 3.8

(i) Every $C_{\cdot 0}$*-representation is c.n.c.*

(ii) A representation is c.n.c if and only if $P_\infty(K) = \overline{P_\infty(L_\infty(\mathcal{D}))}$, *which happens if and only if* $P_\infty(H) \subseteq \overline{P_\infty(L_\infty(\mathcal{D}))}$.

We record here for the sake of reference the following statement, which is part of Theorem 7.3 of [28].

Theorem 3.9 *If (T, σ) is a completely contractive covariant representation of a W^*-correspondence on a Hilbert space H, and if (T, σ) is completely non-coisometric, then $T \times \sigma$ extends to an ultraweakly continuous, completely contractive representation of the Hardy algebra, $H^\infty(E)$, on H.*

3.3 Characteristic Operators

We now turn to the construction of the characteristic operator and the characteristic function associated to a covariant representation. At the outset, we do not require that the representation is *c.n.c.* We fix a completely contractive covariant representation (T, σ) acting on the Hilbert space H. We maintain the notation just developed. However, we shall write W_∞ for the Hilbert space isomorphism that we would have written W_{K_0} earlier in order to lighten the notation. So W_∞ is a Hilbert space isomorphism from $\mathcal{F}(E) \otimes_\rho K_0$ onto $L_\infty(K_0)$ that satisfies (10) and (11) (with K_0 in place of $\mathcal{M}$). We also write u for the isometry from K_0 onto $\mathcal{D}_*$ described in Lemma 3.4. It induces an isometry, written $I_{\mathcal{F}(E)} \otimes u$ from $\mathcal{F}(E) \otimes K_0$ onto $\mathcal{F}(E) \otimes \mathcal{D}_*$.

Definition 3.10 *Let (T, σ) be a completely contractive covariant representation of the W^*-correspondence E over the W^*-algebra M and let (V, ρ) be the minimal isometric dilation of (T, σ). Also, in the notation just established, let τ_1 be the restriction of ρ to $\mathcal{D}$ and let τ_2 be the restriction of ρ (or σ) to $\mathcal{D}_*$. Then the operator Θ_T defined from $\mathcal{F}(E) \otimes_\rho \mathcal{D}$ to $\mathcal{F}(E) \otimes_\rho \mathcal{D}_*$ by the equation*

$$\Theta_T := (I_{\mathcal{F}(E)} \otimes u) \circ W_\infty^*(I - P_\infty)W_{\mathcal{D}} \tag{12}$$

is called the characteristic operator *of the representation (T, σ).*

Remarks 3.11

(i) Evidently, Θ_T is a contraction. Indeed, since $I_{\mathcal{F}(E)} \otimes u$, W_∞ and $W_{\mathcal{D}}$ are all isometries, the "only" things that keep Θ_T from being an isometry are the relations among the range of W_∞, the range of $I - P_\infty$ and $W_{\mathcal{D}}$. Further, given the calculations involving W_∞, $I - P_\infty$ and $W_{\mathcal{D}}$ that we have made so far, it is clear that Θ_T carries some *information about the location of H in the space of the minimal isometric dilation of (T, σ). Our goal is to show that under the assumption that our representation is c.n.c., it carries all the information and is a complete unitary invariant for the representation (T, σ).*

(ii) We frequently will want to refer to the entire system, $(\Theta_T, \mathcal{D}, \mathcal{D}_*, \tau_1, \tau_2)$, as the characteristic operator for the covariant representation (T, σ).

(iii) By definition, τ_2 is the restriction of σ to $\mathcal{D}_*$. By definition of the minimal isometric dilation of (T, σ), (V, ρ), τ_1 really is the restriction of

$\sigma \circ \varphi$ to $\mathcal{D}$ regarded as the zeroth component in the natural decomposition of $\mathcal{F}(E) \otimes_{\sigma_1} \mathcal{D}$. See Definition 3.1.

(iv) Although Θ_T is defined to be a map between the two Hilbert spaces, $\mathcal{F}(E) \otimes \mathcal{D}$ and $\mathcal{F}(E) \otimes \mathcal{D}_*$, which are different, in general, we shall occasionally identify Θ_T with the 2×2 operator matrix

$$\begin{pmatrix} 0 & 0 \\ \Theta_T & 0 \end{pmatrix}$$

in $B(\mathcal{F}(E) \otimes (\mathcal{D} \oplus \mathcal{D}_*))$.

Several basic properties of Θ_T are established in the following lemma.

Lemma 3.12 *The characteristic operator Θ_T is a contraction that satisfies the equations*

$$(\varphi_\infty(a) \otimes I_{\mathcal{D}_*})\Theta_T = \Theta_T(\varphi_\infty(a) \otimes I_{\mathcal{D}}), \quad a \in M \tag{13}$$

and

$$\Theta_T(T_\xi \otimes I_{\mathcal{D}}) = (T_\xi \otimes I_{\mathcal{D}_*})\Theta_T, \ \xi \in E. \tag{14}$$

That is, Θ_T intertwines the representations of $H^\infty(E)$ induced by τ_1 and τ_2. Further, if (T, σ) is a $C_{\cdot 0}$-representation, then $Q_\infty = I$, i.e., $P_\infty = 0$, and Θ_T is an isometry from $\mathcal{F}(E) \otimes \mathcal{D}$ into $\mathcal{F}(E) \otimes \mathcal{D}_*$.

Proof. We already have remarked that Θ_T is a contraction. The other parts of the lemma are immediate consequences of equation (10), Lemma 3.4, Lemma 3.3 and the equations $W_{\mathcal{D}}(T_\xi \otimes I_{\mathcal{D}}) = V(\xi)W_{\mathcal{D}}$ and $W_\infty^* V(\xi) = (T_\xi \otimes I_{K_0})W_\infty^*$, which are easy to check. $\square$

As we shall show in Theorem 3.19, there is a conditioned converse to the last assertion in Lemma 3.12.

The representations τ_1 and τ_2, defined above, need not be faithful. Indeed, they need not even be jointly faithful. This will have to be accommodated in our analysis. Accordingly, we let e be the central projection in M such that $Ker(\tau_1 \oplus \tau_2) = eM$. The following lemma reveals its significance.

Lemma 3.13 *The projection e is the largest central projection q in M such that the operator $\sigma(q)\tilde{T}$ is a partial isometry with initial space $\varphi(q)E \otimes H$ and final space $\sigma(q)H$.*

Proof. The projection e is the largest central projection q with $\tau_1(q) = \tau_2(q) = 0$. But this holds if and only if both the restriction of $\sigma(q)$ to $\Delta_* H$ and the restriction of $\varphi(q) \otimes I_H$ to $\Delta(E \otimes H)$ are equal to zero. This is equivalent to the requirements that $\sigma(q)(I_H - \tilde{T}\tilde{T}^*) = 0$ and $(\varphi(q) \otimes I)(I_{E\otimes H} - \tilde{T}^*\tilde{T}) = 0$. Since $\sigma(q)\tilde{T} = \tilde{T}(\varphi(q) \otimes I)$, the proof is complete. $\square$

Corollary 3.14 *If either $\|\tilde{T}\| < 1$ or M is a factor, then $\tau_1 \oplus \tau_2$ is faithful and $e = 0$.*

3.4 Characteristic Functions

The technology involving the theory of duality that was developed in [28], and is summarized in Section 2, requires faithful representations of the W^*-algebras in question. Since $\tau_1 \oplus \tau_2$ need not be faithful, we will "supplement" it to build a faithful representation of M. For this purpose, we introduce the following terminology.

Definition 3.15 *For $i = 1, 2$, let $\tau_i : M \to B(\mathcal{E}_i)$ be a normal representation of M on $\mathcal{E}_i$ and let e be the central projection such that $\ker(\tau_1 \oplus \tau_2) = eM$. Chose a faithful representation π_0 of M on a Hilbert space H_0 and let τ_0 be the representation of M on $\pi_0(e)H_0$ obtained by restricting π_0 to eM. Form the Hilbert space $\mathcal{E} := \pi_0(H_0) \oplus \mathcal{E}_1 \oplus \mathcal{E}_2$ and let $\tau := \tau_0 \oplus \tau_1 \oplus \tau_2$ be the (necessarily faithful) representation of M on $\mathcal{E}$. Then we call $\mathcal{E}$ a* supplemental space *for the pair of representations τ_1 and τ_2, we shall call the representation τ of M on $\mathcal{E}$ a* supplemental representation *and we shall simply call the pair $(\mathcal{E}, \tau)$ a* supplement *for τ_1 and τ_2.*

Evidently, if τ_1 and τ_2 are jointly faithful, then $(\mathcal{E}_1 \oplus \mathcal{E}_2, \tau_1 \oplus \tau_2)$ is the only possible supplement for τ_1 and τ_2. We shall see shortly that the use of supplemental spaces and representations is a matter of convenience only and that the constructs we consider do not depend in any material way on the choice of π_0 used to define them.

Suppose, now, that $(\Theta_T, \mathcal{D}, \mathcal{D}_*, \tau_1, \tau_2)$ is the characteristic operator determined by a covariant representation (T, σ) of E. We fix once and for all a supplement $(\mathcal{G}, \tau)$ for τ_1 and τ_2 and we consider $\mathcal{F}(E) \otimes_\tau \mathcal{G}$ as written as the direct sum

$$\mathcal{F}(E) \otimes_\tau \mathcal{G} = (\mathcal{F}(E) \otimes_{\pi_0} H_0) \oplus (\mathcal{F}(E) \otimes_{\tau_1} \mathcal{D}) \oplus (\mathcal{F}(E) \otimes_{\tau_2} \mathcal{D}_*). \tag{15}$$

Corresponding to this direct sum decomposition of $\mathcal{F}(E) \otimes_\sigma \mathcal{G}$, we shall identify Θ_T with the block matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Theta_T & 0 \end{pmatrix}. \tag{16}$$

Since Θ_T satisfies equations (14) and (13), it follows that this block matrix actually lies in the commutant of $\tau^{\mathcal{F}(E)}(H^\infty(E))$. Hence we may take its Fourier transform relative to τ as in Remark 2.11, obtaining an element $\hat{\Theta}_T \in H^\infty(E^\tau)$ such that

$$\hat{\Theta}_T \otimes I_{\mathcal{G}} = U\Theta_T U^*, \tag{17}$$

where U is the Fourier transform from $\mathcal{F}(E) \otimes_\tau \mathcal{G}$ onto $\mathcal{F}(E^\tau) \otimes_\iota \mathcal{G}$ defined in Proposition 2.9. Since elements of $H^\infty(E^\tau)$ may be viewed as functions on the unit ball of E (see Remark 2.11), we will think of $\hat{\Theta}_T$ as being so represented when we wish. The following lemma records some of the properties of this transform and shows that it does not really depend on the choice of τ and $\mathcal{G}$.

Lemma 3.16 *Let $\hat{\Theta}_T$ be the element of $H^\infty(E^\tau)$ defined in equation (17) using the Fourier transform U from $\mathcal{F}(E) \otimes_\tau \mathcal{G}$ onto $\mathcal{F}(E^\tau) \otimes_\iota \mathcal{G}$. Also let q_1 be the projection from $\mathcal{G}$ onto $\mathcal{D}$ and q_2 be the projection onto $\mathcal{D}_*$. Then both q_1 and q_2 lie in $\tau(M)'$, and*

(i) $U^*(q_i \otimes I_{\mathcal{G}})U = I_{\mathcal{F}(E)} \otimes q_i$, $i = 1, 2$.

(ii) $\hat{\Theta}_T = q_2 \hat{\Theta}_T q_1$ and, if (T, σ) is a $C_{\cdot 0}$-representation, then $\hat{\Theta}_T^* \hat{\Theta}_T = q_1$.

(iii) For every $\xi \in E$ with $\|\xi\| < 1$, $q_2 \hat{\Theta}_T(\xi) q_1 = \hat{\Theta}_T(\xi)$.

Proof. To prove (i), recall first that, for $\eta_1, \ldots, \eta_k$ in E^τ and $h \in \mathcal{G}$,

$$U^*(\eta_1 \otimes \ldots \otimes \eta_k \otimes h) = (I_{E^{\otimes(k-1)}} \otimes \eta_1) \cdots (I_E \otimes \eta_{k-1})\eta_k(h).$$

For $q \in \tau(M)'$ and $\eta \in E^\tau$, we have $q \cdot \eta = (I_E \otimes q)\eta$. (This is the left action of $\tau(M)'$ on E^τ.) Thus, for such q,

$$U^*(q \otimes I_{\mathcal{G}})(\eta_1 \otimes \ldots \otimes \eta_k \otimes h) = U^*(q\eta_1 \otimes \ldots \otimes \eta_k \otimes h) =$$

$$= (I_{E^{\otimes k}} \otimes q)(I_{E^{\otimes(k-1)}} \otimes \eta_1) \cdots (I_E \otimes \eta_{k-1})\eta_k(h) = (I_{E^{\otimes k}} \otimes q)U^*(\eta_1 \otimes \ldots \otimes \eta_k \otimes h).$$

This proves (i). From the construction of the operator Θ_T above it follows that $\Theta_T = (I_{\mathcal{F}(E)} \otimes q_2)\Theta_T(I_{\mathcal{F}(E)} \otimes q_1)$. Thus, using (i), $U\Theta_T U^* = U(I_{\mathcal{F}(E)} \otimes q_2)U^*U\Theta_T U^*U(I_{\mathcal{F}(E)} \otimes q_1)U^* = (q_2 \otimes I_{\mathcal{G}})U\Theta_T U^*(q_1 \otimes I_{\mathcal{G}})$. Since $\hat{\Theta}_T \otimes I_{\mathcal{G}} = U\Theta_T U^*$, we proved (ii). For $X \in H^\infty(E^\tau)$, $X(\xi)$ is the image, under a certain representation of $H^\infty(E^\tau)$ defined by ξ, of X. Thus the map $X \mapsto X(\xi)$ is multiplicative and it carries elements of $\tau(M)'$ to themselves. Part (iii) thus follows from part (ii). $\square$

The lemma shows that $q_2\hat{\Theta}_T(\xi)q_1 = \hat{\Theta}_T(\xi)$ for all ξ in the open unit ball of E and so we may view $\hat{\Theta}_T$ as a function from the open unit ball of E to $B(\mathcal{D}, \mathcal{D}_*)$.

The properties of $\hat{\Theta}_T$ will be formalized in the following definition.

Definition 3.17 *Given a W^*-algebra M and a W^*-correspondence E over M, a* characteristic function *is a system $(\Theta, \mathcal{E}_1, \mathcal{E}_2, \tau_1, \tau_2)$ with the following properties:*

(i) For $i = 1, 2$, $\mathcal{E}_i$ is a Hilbert space and τ_i is a representation of M on $\mathcal{E}_i$.

(ii) If $(\mathcal{E}, \tau)$ is a supplement for τ_1 and τ_2, and if q_i is the projection of $\mathcal{E}$ onto $\mathcal{E}_i$, $i = 1, 2$, then Θ is a contraction in $H^\infty(E^\tau)$ satisfying $\Theta = q_2\Theta q_1$.

If, in addition, Θ satisfies $\Theta^*\Theta = q_1$ then Θ will be called an inner characteristic function.

Very often we shall write Θ for the tuple $(\Theta, \mathcal{E}_1, \mathcal{E}_2, \tau_1, \tau_2)$. Also, given a characteristic function, we shall freely use the notation set in Definition 3.17 (i.e. $\mathcal{E}_i$, τ_i and q_i).

Definition 3.18 *If (T, σ) is a covariant representation of the W^*-correspondence E, then the system $(\hat{\Theta}_T, \mathcal{D}, \mathcal{D}_*, \tau_1, \tau_2)$ defined by equation (17), or simply $\hat{\Theta}_T$, will be called* the characteristic function of the representation (T, σ).

The following result is familiar from the theory of single operators. It is the "converse" of Lemma 3.12.

Theorem 3.19 *Let E be a W^*-correspondence over a W^*-algebra M and let (T, σ) be a c.n.c. representation of E on the Hilbert space H. Then the characteristic function $\hat{\Theta}_T$ of the covariant representation (T, σ) is inner if and only if (T, σ) is a $C_{\cdot 0}$-representation.*

Proof. Lemma 3.12 shows that if (T,σ) is a $C_{\cdot 0}$ representation, then Θ_T is an isometry. Consequently, $\hat{\Theta}_T$ is inner. To prove the converse, observe that from the definition of Θ_T, equation (12), Θ_T is an isometry if and only if $L_\infty(\mathcal{D}) \subseteq L_\infty(K_0)$. However, by our assumption that (T,σ) is c.n.c., we know by definition (Definition 3.5) that $L_\infty(\mathcal{D}) \vee L_\infty(K_0) = K$. Hence, if Θ_T is an isometry, so that $L_\infty(\mathcal{D}) \subseteq L_\infty(K_0)$, we conclude that $L_\infty(K_0) = K$. Hence by definition (Definition 3.5), (T,σ) is a $C_{\cdot 0}$ representation. Since Θ_T is an isometry if and only if $\hat{\Theta}_T$ is inner, the proof is complete. $\square$

3.5 Pointwise Evaluations

Of course several natural questions arise at this point: Is every characteristic function the characteristic function of some representation? If so, how is the representation constructed? What is the level of uniqueness among the constructs? Before tackling these, we first compute the values $\hat{\Theta}_T(\xi)$ for the characteristic function of a covariant representation (T,σ). The calculations will play roles in the sequel. The initial step of our analysis is the following computation.

Lemma 3.20 *Let $P_{\mathcal{D}}$ (resp. $P_{\mathcal{D}_*}$) denote the projection of $\mathcal{F}(E) \otimes_\tau \mathcal{D} = \mathcal{D} \oplus (E \otimes_\tau \mathcal{D}) \oplus \cdots$ onto the zeroth summand, $\mathcal{D}$ (resp. the projection of $\mathcal{F}(E) \otimes_\tau \mathcal{D}_*$ onto the zeroth summand $\mathcal{D}_*$). Also, for $\xi \in E$, $\|\xi\| \leq 1$, write $L_{\xi^{\otimes k}}$ for the operator from $\mathcal{F}(E) \otimes \mathcal{D}_*$ to $\mathcal{F}(E) \otimes \mathcal{D}_*$ defined by formula $L_{\xi^{\otimes k}} \eta \otimes h = \xi^{\otimes k} \otimes \eta \otimes h$, when $k \geq 1$, and let $L_{\xi^{\otimes 0}}$ be the identity operator. Then for every ξ in the open unit ball of E and every $g \in \mathcal{D}$*

$$\hat{\Theta}_T(\xi) g = \sum_{k=0}^{\infty} P_{\mathcal{D}} L^*_{\xi^{\otimes k}} (I_{\mathcal{F}(E)} \otimes u) W^*_\infty Q_\infty g,$$

where, recall, $W_\infty : \mathcal{F}(E) \otimes K_0 \to K$ and $u : K_0 \to \mathcal{D}_*$ are the isometries defined above.

Proof. Note first that, since $\|\xi\| < 1$, the sum converges. To establish the formula we shall fix such a ξ and show that for every $R \in H^\infty(E^\tau)$ and every $g \in \mathcal{D}$,

$$R(\xi) g = \sum_{k=0}^{\infty} P_{\mathcal{G}} L^*_{\xi^{\otimes k}} U^* (R \otimes I_{\mathcal{G}}) U g \tag{18}$$

where, recall, $\mathcal{G}$ is $\pi_0(e)H_0 \oplus \mathcal{D} \oplus \mathcal{D}_*$ and U is the Fourier transform from $\mathcal{F}(E) \otimes_\tau \mathcal{G}$ to $\mathcal{F}(E^\tau) \otimes_\iota \mathcal{G}$, while $P_\mathcal{G}$ is the projection of $\mathcal{F}(E) \otimes_\tau \mathcal{G}$ onto the zeroth summand. When $R = \hat{\Theta}_T$ we will obtain the desired result since $U^*(\hat{\Theta}_T \otimes I_\mathcal{G})U = \Theta_T$. Suppose first that $R = \varphi_\infty(b) \in H^\infty(E^\tau)$ (with $b \in M$). Then $R(\xi) = b$ by definition. Computing the right hand side of (18) we get first $U^*(\varphi_\infty(b) \otimes I_\mathcal{G})Ug = U^*bg = bg$ and, thus, the only non zero term in the sum is the one corresponding to $k = 0$. In this event the sum is then equal to bg, proving the equation for constant functions. Now fix $m \geq 1$, let $\eta = \eta_1 \otimes \eta_2 \cdots \otimes \eta_m \in (E^\tau)^{\otimes m}$ and let $R = T_\eta \in H^\infty(E^\tau)$. Then, from the definition of $R(\xi)$,

$$R(\xi) = (T_{\eta_1})(\xi) \cdots (T_{\eta_m})(\xi) = (L_\xi^* \eta_1) \cdots (L_\xi^* \eta_m)$$

where η_i is viewed as a map from $\mathcal{G}$ into $E \otimes_\tau \mathcal{G}$ and, thus, $L_\xi^* \eta_i \in B(\mathcal{G})$.

To compute the right hand side of (18) in this case we first compute $U^*(T_\eta \otimes I_\mathcal{G})Ug = U^*(\eta \otimes g) = (I_{(E^\tau)^{\otimes(m-1)}} \otimes \eta_1) \cdots (I_{E^\tau} \otimes \eta_{m-1})\eta_m(g)$. It then follows that the only non zero term in the sum is the one that corresponds to $k = m$. A simple computation shows that

$$L^*_{\xi^{\otimes m}}(I_{(E^\tau)^{\otimes(m-1)}} \otimes \eta_1) \cdots (I_{E^\tau} \otimes \eta_{m-1})\eta_m(g) = (L_\xi^* \eta_1) \cdots (L_\xi^* \eta_m)g.$$

This, by linearity, proves (18) for a σ-weakly dense subset of $H^\infty(E^\tau)$. Since both sides of the equation are σ-weakly continuous (as a function of R), equation (18) follows. $\square$

To use lemma 3.20 to calculate the values of $\hat{\Theta}_T(\xi)$, we compute the series appearing in the lemma term by term. For $k = 0$ we have $P_{\mathcal{D}_*}\Theta_T g = P_{\mathcal{D}_*} u W_\infty^* Q_\infty g = uQ_0 g$ for all $g \in \mathcal{D}$. Suppose $g = \Delta(\theta \otimes h)$, $\theta \otimes h \in E \otimes_\tau H$. Then

$$uQ_0 g = uQ_0 \Delta(\theta \otimes h) = u(-\tilde{T}\Delta^2(\theta \otimes h) + (I_\mathcal{D} - \Delta^2)\Delta(\theta \otimes h)) =$$

$$= u(-\Delta_*^2 \tilde{T}(\theta \otimes h) + \Delta \tilde{T}^* \tilde{T}(\theta \otimes h)) = -\Delta_* \tilde{T}(\theta \otimes h) = -\tilde{T}\Delta(\theta \otimes h) = -\tilde{T}g.$$

Since vectors g of the form $\Delta(\theta \otimes h)$ generate $\mathcal{D}$, we see that

$$uQ_0|\mathcal{D} = -\tilde{T}|\mathcal{D} \tag{19}$$

and, thus, the zeroth term in the expression of $\Theta(\xi)$ is $-\tilde{T}$. To compute the other terms recall first, from equation (8), that we can write $\tilde{V}^*$ matricially

as

$$\tilde{V}^* = \begin{pmatrix} \tilde{T}^* & \Delta & 0 & \dots \\ 0 & 0 & I & \\ 0 & 0 & 0 & \\ & & & \ddots \end{pmatrix} : H \oplus \mathcal{D} \oplus \cdots \to E \otimes H \oplus E \otimes \mathcal{D} \oplus \cdots \qquad (20)$$

Thus, for $g \in \mathcal{D}$, $\tilde{V}^* g = \Delta g$ and $\tilde{V}_2^* g = (I_E \otimes \tilde{V}^*)\tilde{V}^* g = (I_E \otimes \tilde{T}^*)\Delta g$. In fact, for every $k \geq 2$,

$$\tilde{V}_k^* g = (I_{E^{\otimes(k-1)}} \otimes \tilde{T}^*) \cdots (I_E \otimes \tilde{T}^*)\Delta g$$

for $g \in \mathcal{D}$.

The next term ($k = 1$) applied to $g = \Delta(\theta \otimes h)$ is

$$L_\xi^*(I_E \otimes u)W_\infty^* Q_\infty \Delta(\theta \otimes h) = L_\xi^*(I_E \otimes u)W_\infty^* \tilde{V}(I_E \otimes Q_0)\tilde{V}^* \Delta(\theta \otimes h) =$$

$$= L_\xi^*(I_E \otimes uQ_0)\tilde{V}^*\Delta(\theta \otimes h).$$

Using the comments above, $\tilde{V}^*\Delta(\theta \otimes h) = \Delta^2(\theta \otimes h)$. Also, for $h \in H$, we have

$$uQ_0 h = u(\Delta_*^2 h \oplus (-\Delta\tilde{T}^* h)) = \Delta_* h,$$

by lemma 3.4. Hence $L_\xi^*(I_E \otimes uQ_0)V_\infty^*\Delta(\theta \otimes h) = L_\xi^*(I_E \otimes \Delta_*)\Delta^2(\theta \otimes h) = \Delta_* L_\xi^* \Delta^2(\theta \otimes h)$. It follows that the term that corresponds to $k = 1$ in the expression of $\hat{\Theta}_T(\xi)$ is $\Delta_* L_\xi^* \Delta$. Continuing in this fashion, we see that for $k \geq 2$ and $g = \Delta(\theta \otimes h)$, we have

$$L_{\xi^{\otimes k}}^*(I_{E^{\otimes k}} \otimes u)W_\infty^* \tilde{V}_k(I_{E^{\otimes k}} \otimes Q_0)\tilde{V}_k^* g = L_{\xi^{\otimes k}}^*(I_{E^{\otimes k}} \otimes uQ_0)\tilde{V}_k^*\Delta(\theta \otimes h) =$$

$$= L_{\xi^{\otimes k}}^*(I_{E^{\otimes k}} \otimes \Delta_*)(I_{E^{\otimes(k-1)}} \otimes \tilde{T}^*) \cdots (I_E \otimes \tilde{T}^*)\Delta g = \Delta_*(L_\xi^* \tilde{T}^*)^{k-1} L_\xi^* \Delta g.$$

Thus the kth term in the expression of $\hat{\Theta}_T(\xi)$ is $\Delta_*(L_\xi^*\tilde{T}^*)^{k-1}L_\xi^*\Delta$. We now summarize the discussion above.

Theorem 3.21 *The values of the characteristic function $\hat{\Theta}_T$ on the open unit ball of E can be written as*

$$\hat{\Theta}_T(\xi) = -\tilde{T}|\mathcal{D} + \sum_{k=1}^{\infty} \Delta_*(L_\xi^*\tilde{T}^*)^{k-1} L_\xi^* \Delta|\mathcal{D} = -\tilde{T}|\mathcal{D} + \Delta_*(I - L_\xi^*\tilde{T})^{-1} L_\xi^* \Delta|\mathcal{D}.$$

Remark 3.22 *Theorem 3.21 may be viewed as a realization formula associated with the unitary operator matrix*

$$\begin{pmatrix} -\tilde{T}|\mathcal{D} & \Delta_* \\ \Delta|\mathcal{D} & \tilde{T}^* \end{pmatrix} : \mathcal{D} \oplus H \to \mathcal{D}_* \oplus (E \otimes_\sigma H).$$

(See e.g. [1] .) Evidently, it is an exact analogue of the formula for the characteristic operator function for a single contraction operator [41].

3.6 Models from Characteristic Functions

Suppose we are given a characteristic function $(\Theta, \mathcal{E}_1, \mathcal{E}_2, \tau_1, \tau_2)$ and form $\check{\Theta} := U^*(\Theta \otimes I_{\mathcal{E}})U$ where, recall, $\mathcal{E} := \pi_0(e)H \oplus \mathcal{E}_1 \oplus \mathcal{E}_2$ is the Hilbert space described in Definition 3.17 and $U : \mathcal{F}(E) \otimes_\tau \mathcal{E} \to \mathcal{F}(E^\tau) \otimes_\iota \mathcal{E}$ is the Fourier transform described in Proposition 2.9 and Remark 2.11. Then $\check{\Theta}$ commutes with the operators $T_\xi \otimes I_{\mathcal{E}}$ and $\varphi_\infty(a) \otimes I_{\mathcal{E}}$ for $\xi \in E$ and $a \in M$. Since $\Theta = q_2 \Theta q_1$, we can use the argument of the proof of Lemma 3.16 (i) to show that $U^*(q_i \otimes I_{\mathcal{E}})U = I_{\mathcal{F}(E)} \otimes q_i$, $i = 1, 2$, and, thus, $\check{\Theta}(\mathcal{F}(E) \otimes \mathcal{E}_1) = U^*(\Theta \mathcal{F}(E^\tau) \otimes \mathcal{E}) = U^*(q_2 \otimes I)(\Theta \mathcal{F}(E^\tau) \otimes \mathcal{E}) \subseteq \mathcal{F}(E) \otimes \mathcal{E}_2$. It follows that, for $\xi \in E$ and $a \in M$,

$$\check{\Theta}(T_\xi \otimes I_{\mathcal{E}_1}) = (T_\xi \otimes I_{\mathcal{E}_2})\check{\Theta} \tag{21}$$

and

$$\check{\Theta}(\varphi_\infty(a) \otimes I_{\mathcal{E}_1}) = (\varphi_\infty(a) \otimes I_{\mathcal{E}_2})\check{\Theta} \tag{22}$$

Our objective is to show that there is a covariant representation (T, σ) of E such that $\Theta_T = \check{\Theta}$. To this end, we write $\Delta_{\check{\Theta}} = (I_{\mathcal{F}(E) \otimes \mathcal{E}_1} - \check{\Theta}^* \check{\Theta})^{1/2} \in B(\mathcal{F}(E) \otimes_{\tau_1} \mathcal{E}_1)$ and set

$$K(\Theta) := (\mathcal{F}(E) \otimes_{\tau_2} \mathcal{E}_2) \oplus \overline{\Delta_{\check{\Theta}}(\mathcal{F}(E) \otimes_{\tau_1} \mathcal{E}_1)} \subseteq \mathcal{F}(E) \otimes_\tau \mathcal{E} \tag{23}$$

and

$$H(\Theta) := ((\mathcal{F}(E) \otimes_{\tau_2} \mathcal{E}_2) \oplus \overline{\Delta_{\check{\Theta}}(\mathcal{F}(E) \otimes_{\tau_1} \mathcal{E}_1)}) \ominus \{\check{\Theta}\xi \oplus \Delta_{\check{\Theta}}\xi \mid \xi \in \mathcal{F}(E) \otimes_{\tau_1} \mathcal{E}_1\}. \tag{24}$$

Note that if Θ is inner, then $\check{\Theta}^* \check{\Theta} = U^*(q_1 \otimes I_{\mathcal{E}})U = I_{\mathcal{F}(E)} \otimes q_1$ and so $\Delta_{\check{\Theta}} = 0$. Thus, in this case $K(\Theta) = \mathcal{F}(E) \otimes_{\tau_2} \mathcal{E}_2$ and $H(\Theta) = (\mathcal{F}(E) \otimes_{\tau_2} \mathcal{E}_2) \ominus \check{\Theta}(\mathcal{F}(E) \otimes_{\tau_1} \mathcal{E}_1)$.

We shall also write P_Θ for the projection from $K(\Theta)$ onto $H(\Theta)$.

Lemma 3.23 *Let Θ be a characteristic function and let $\check{\Theta}$, $K(\Theta)$ and $H(\Theta)$ be the operator and spaces just defined. For every $a \in M$ and $\xi \in E$ we define the operators $S_\Theta(\xi)$ and $\psi_\Theta(a)$ on $\Delta_{\check{\Theta}}(\mathcal{F}(E) \otimes \mathcal{E}_1)$ by the formulae*

$$S_\Theta(\xi)\Delta_{\check{\Theta}} g = \Delta_{\check{\Theta}}(T_\xi \otimes I_{\mathcal{E}_1})g, \quad g \in \mathcal{F}(E) \otimes \mathcal{E}_1 \tag{25}$$

and

$$\psi_\Theta(a)\Delta_{\check{\Theta}} g = \Delta_{\check{\Theta}}(\varphi_\infty(a) \otimes I_{\mathcal{E}_1})g, \quad g \in \mathcal{F}(E) \otimes \mathcal{E}_1. \tag{26}$$

Also, we define the following operators on $K(\Theta)$:

$$V_\Theta(\xi) = (T_\xi \otimes I_{\mathcal{E}_2}) \oplus S_\Theta(\xi) \tag{27}$$

and

$$\rho_\Theta(a) = (\varphi_\infty(a) \otimes I_{\mathcal{E}_2}) \oplus \psi_\Theta(a). \tag{28}$$

Then

(i) (S_Θ, ψ_Θ) and (V_Θ, ρ_Θ) are isometric covariant representations of E on $\Delta_{\check{\Theta}}(\mathcal{F}(E) \otimes \mathcal{E}_1)$ and $K(\Theta)$ respectively.

(ii) The space $K(\Theta) \ominus H(\Theta)$ is invariant for (V_Θ, ρ_Θ) and, thus, the compression of (V_Θ, ρ_Θ) to $H(\Theta)$, which we denote by $(T_\Theta, \sigma_\Theta)$, is a completely contractive covariant representation of E. Explicitly,

$$T_\Theta(\xi) = P_\Theta V_\Theta(\xi)|H(\Theta), \quad \xi \in E \tag{29}$$

and

$$\sigma_\Theta(a) = P_\Theta \rho_\Theta(a)|H(\Theta), \quad a \in M. \tag{30}$$

Proof. In (i) it is enough to prove the statement about (S_Θ, ψ_Θ). We shall write Δ for $\Delta_{\check{\Theta}}$. Then, for $\xi \in E$, $a, b \in M$ and $g \in \mathcal{F}(E) \otimes \mathcal{E}_1$, $S_\Theta(a\xi b)\Delta g = \Delta(T_{a\xi b} \otimes I_{\mathcal{E}_1})g = \Delta(\varphi_\infty(a)T_\xi\varphi_\infty(b) \otimes I_{\mathcal{E}_1})g = \psi_\Theta(a)S_\Theta(\xi)\psi_\Theta(b)\Delta g$. This proves the covariance property. Since $(\varphi_\infty(a) \otimes I_{\mathcal{E}_2})\check{\Theta} = \check{\Theta}(\varphi_\infty(a) \otimes I_{\mathcal{E}_1})$, $\varphi_\infty(a) \otimes I_{\mathcal{E}_1}$ commutes with Δ and ψ_Θ is a $*$-representation of M. To show that the representation is isometric we compute for $\eta_i \otimes h_i \in \mathcal{F}(E) \otimes \mathcal{E}_1$, $i = 1$

and 2,

$$
\begin{aligned}
&\langle S_\Theta(\xi_1)\Delta(\eta_1\otimes h_1), S_\Theta(\xi_2)\Delta(\eta_2\otimes h_2)\rangle \\
&\qquad = \langle \Delta(\xi_1\otimes\eta_1\otimes h_1), \Delta(\xi_2\otimes\eta_2\otimes h_2\rangle \\
&\qquad = \langle \xi_1\otimes\eta_1\otimes h_1, \xi_2\otimes\eta_2\otimes h_2\rangle - \langle \check\Theta(\xi_1\otimes\eta_1\otimes h_1), \check\Theta(\xi_2\otimes\eta_2\otimes h_2)\rangle \\
&\qquad = \langle \eta_1\otimes h_1, \varphi_\infty(\langle\xi_1,\xi_2\rangle)\eta_2\otimes h_2\rangle - \langle \xi_1\otimes\check\Theta(\eta_1\otimes h_1), \xi_2\otimes\check\Theta(\eta_2\otimes h_2)\rangle \\
&\qquad = \langle \eta_1\otimes h_1, \varphi_\infty(\langle\xi_1,\xi_2\rangle)\eta_2\otimes h_2\rangle - \langle \check\Theta(\eta_1\otimes h_1), (\varphi_\infty(\langle\xi_1,\xi_2\rangle)\otimes I_{\mathcal{E}_2})\check\Theta(\eta_2\otimes h_2)\rangle \\
&\qquad = \langle \eta_1\otimes h_1, \varphi_\infty(\langle\xi_1,\xi_2\rangle)\eta_2\otimes h_2\rangle - \langle \check\Theta(\eta_1\otimes h_1), \check\Theta(\varphi_\infty(\langle\xi_1,\xi_2\rangle)\otimes I_{\mathcal{E}_1})(\eta_2\otimes h_2)\rangle \\
&\qquad = \langle \Delta^2(\eta_1\otimes h_1), (\varphi_\infty(\langle\xi_1,\xi_2\rangle)\otimes I_{\mathcal{E}_1})(\eta_2\otimes h_2)\rangle \\
&\qquad = \langle \Delta(\eta_1\otimes h_1), \psi_\Theta(\langle\xi_1,\xi_2\rangle)\Delta(\eta_2\otimes h_2)\rangle.
\end{aligned}
$$

This shows that the representation is isometric. To prove (ii) all we have to show is the invariance of $K(\Theta)\ominus H(\Theta) = \{\check\Theta g \oplus \Delta g : g \in \mathcal{F}(E)\otimes\mathcal{E}_1\}$ under the representation (V_Θ, ρ_Θ). However, this is an immediate application of equations (21) and (22). □

Definition 3.24 *Let $(\Theta, \mathcal{E}_1, \mathcal{E}_2, \tau_1, \tau_2)$ be a characteristic function. Then the covariant representation $(T_\Theta, \sigma_\Theta)$ on $H(\Theta)$ defined from Θ in Lemma 3.23 is called the* canonical model constructed from Θ. *If $(\Theta, \mathcal{E}_1, \mathcal{E}_2, \tau_1, \tau_2)$ is the characteristic function of a covariant representation (T, σ), i.e., if $\Theta = \hat\Theta_T$, then $(T_\Theta, \sigma_\Theta)$ will be called the* canonical model for (T, σ).

We begin to justify this terminology in the following Theorem.

Theorem 3.25 *Let (T, σ) be a c.n.c. covariant representation of E, with characteristic operator Θ_T. Let $\Theta := \hat\Theta_T$ be the associated characteristic function and $(T_\Theta, \sigma_\Theta)$ be the canonical model for (T, σ). Then (T, σ) and $(T_\Theta, \sigma_\Theta)$ are unitarily equivalent.*

Proof. Let H be the representation space of (T, σ) and recall the definition of Θ_T in Definition 12. Note that in the notation of Lemma 3.23, $\Theta_T = \check\Theta$. Write

$$\Phi_1 = W_\infty(I_{\mathcal{F}(E)}\otimes u^*) : \mathcal{F}(E)\otimes\mathcal{D}_* \to K$$

where K and $\mathcal{D}_*$ are the spaces associated with (T, σ) and its minimal isometric dilation, and where W_∞ and u are the operators defined in the discussion

preceding Definition 12. Then Φ_1 is an isometry whose range is $L_\infty(K_0)$. We also define $\Phi_2 : \Delta_{\check{\Theta}}(\mathcal{F}(E)\otimes\mathcal{D}) \to P_\infty(K)$ by the equation

$$\Phi_2(\Delta_{\check{\Theta}}x) = P_\infty(W_{\mathcal{D}}x) \ , \quad x \in \mathcal{F}(E)\otimes\mathcal{D}.$$

Since the representation is c.n.c., $P_\infty(L_\infty(\mathcal{D})) = P_\infty(K)$ by part (ii) of Lemma 3.8 and so Φ_2 is surjective. We show that it is an isometry. For this we compute

$$\|\Delta_{\check{\Theta}}\xi\|^2 = \langle (I-\check{\Theta}^*\check{\Theta})\xi,\xi\rangle = \|\xi\|^2 - \|(I_{\mathcal{F}(E)}\otimes u_*)\check{\Theta}\xi\|^2.$$

By definition of $\check{\Theta} = \Theta_T$ (equation (12)), the last expression is equal to

$$\|\xi\|^2 - \|W_\infty^* Q_\infty W_{\mathcal{D}}\xi\|^2 = \|W_{\mathcal{D}}\xi\|^2 - \|Q_\infty W_{\mathcal{D}}\xi\|^2 = \|P_\infty W_{\mathcal{D}}\xi\|^2.$$

Thus Φ_2 is a unitary operator onto $P_\infty(K)$. Setting $\Phi = \Phi_1 \oplus \Phi_2$ we obtain a unitary operator from $K(\Theta)$ onto K.

Next we show that Φ maps $H(\Theta)$ onto H. Fix $x \in \mathcal{F}(E)\otimes\mathcal{D}$. Then by definition,

$$\Phi(\check{\Theta}x \oplus \Delta_{\check{\Theta}}x) = W_\infty(I_{\mathcal{F}(E)}\otimes u^*)\check{\Theta}x + P_\infty(W_{\mathcal{D}}x).$$

So, if $x \in \mathcal{D}$, with $\mathcal{D}$ regarded as the zeroth summand of $\mathcal{F}(E)\otimes\mathcal{D}$, we find from the definition of $\check{\Theta} = \Theta_T$ (equation (12)) that $\Phi(\check{\Theta}x \oplus \Delta_{\check{\Theta}}x) = W_\infty(I_{\mathcal{F}(E)}\otimes u^*)\check{\Theta}x + P_\infty(W_{\mathcal{D}}x) = Q_\infty x + P_\infty x = x$. Since $\mathcal{D}$ is orthogonal to H, we see that $\Phi(\check{\Theta}x \oplus \Delta_{\check{\Theta}}x) \in H^\perp$. If $n \geq 1$, then for $x = \xi\otimes d \in E^{\otimes n}\otimes\mathcal{D}$, we also have

$$\begin{aligned} W_\infty(I_{\mathcal{F}(E)}\otimes u^*)\check{\Theta}x + P_\infty(W_{\mathcal{D}}x) &= W_\infty(I_{\mathcal{F}(E)}\otimes u^*)(\xi\otimes\check{\Theta}d) + P_\infty(V_n(\xi)d) \\ &= V_n(\xi)Q_\infty d + V_n(\xi)P_\infty d = V_n(\xi)d \in H^\perp. \end{aligned}$$

Thus, we find that $\Phi(K(\Theta)\ominus H(\Theta)) = \sum^\oplus V_n(E^{\otimes n})\mathcal{D} = K \ominus H$, and it follows that Φ maps $H(\Theta)$ onto H.

Notice also that for $\xi \in E$

$$\Phi_1(T_\xi \otimes I) = W_\infty(T_\xi\otimes u^*) = V(\xi)W_\infty(I\otimes u^*) = V(\xi)\Phi_1, \tag{31}$$

while

$$\begin{aligned} \Phi_2(S(\xi)\Delta_{\check{\Theta}}x) &= \Phi_2(\Delta_{\check{\Theta}}(T_\xi\otimes I)x) = P_\infty(W_{\mathcal{D}}(T_\xi\otimes I)x) \\ &= P_\infty(V(\xi)W_{\mathcal{D}}x) = V(\xi)P_\infty(W_{\mathcal{D}}x) = V(\xi)\Phi_2(\Delta_{\check{\Theta}}x). \end{aligned}$$

Thus Φ intertwines V and V_Θ. To show that Φ also intertwines ρ and ρ_Θ, we let $a \in M$ and compute:

$$\begin{aligned}\Phi_1(\varphi_\infty(a)\otimes I_{\mathcal{D}_*}) = W_\infty(I_{\mathcal{F}(E)}\otimes u^*)(\varphi_\infty(a)\otimes I) = W_\infty(\varphi_\infty(a)\otimes I)(I_{\mathcal{F}(E)}\otimes u^*) \\ = \rho(a)W_\infty(I_{\mathcal{F}(E)}\otimes u^*),\end{aligned}$$

and, for $x \in \mathcal{F}(E)\otimes\mathcal{D}$,

$$\begin{aligned}\Phi_2(\psi_\Theta(a)(\Delta_{\check{\Theta}}x)) = \Phi_2(\Delta(\varphi_\infty(a)\otimes I_{\mathcal{D}})x) = P_\infty(W_{\mathcal{D}}(\varphi_\infty(a)\otimes I)x) \\ = P_\infty(\rho(a)W_{\mathcal{D}}x) = \rho(a)P_\infty W_{\mathcal{D}}x = \rho(a)\Phi_2(\Delta_{\check{\Theta}}x).\end{aligned}$$

It follows that the restriction of Φ to $H(\Theta)$ gives the desired equivalence. $\square$

Definition 3.26 *Let (T,σ) be a c.n.c. representation of the W^*-correspondence on the Hilbert space H. Let $\Theta := \hat{\Theta}_T$ be the characteristic function for (T,σ) and let (T_Θ,σ_Θ) on $H(\Theta)$ be the canonical model built from Θ. Then the Hilbert space isomorphism Φ from the Hilbert space K of the minimal isometric dilation of (T,σ) to $K(\Theta)$ constructed in the proof of Theorem 3.25 will be called the* canonical (Hilbert space) isomorphism (implementing a unitary equivalence between (T,σ) and (T_Θ,σ_Θ)) *or simply the* canonical equivalence *for short.*

Remark 3.27 *Given a general characteristic function $(\Theta,\mathcal{E}_1,\mathcal{E}_2,\tau_1,\tau_2)$, the isometric representation (V_Θ,ρ_Θ) on $K(\Theta)$ defined by equations (27) and (28) is an isometric dilation of (T_Θ,σ_Θ) by definition. In general, it need not be minimal. However, it will be under hypotheses that we discuss shortly. See Lemma 3.35.*

3.7 Isomorphic Characteristic Functions

Definition 3.28 *Let $(\Theta,\mathcal{E}_1,\mathcal{E}_2,\tau_1,\tau_2)$ and $(\Theta',\mathcal{E}_1',\mathcal{E}_2',\tau_1',\tau_2')$ be two characteristic functions. We say that they are* isomorphic *if there are Hilbert space isomorphisms $W_i : \mathcal{E}_i \to \mathcal{E}_i'$ that intertwine τ_i and τ_i', $i = 1$ and 2, and satisfy the equation*

$$\check{\Theta}' = (I_{\mathcal{F}(E)}\otimes W_2)\check{\Theta}(I_{\mathcal{F}(E)}\otimes W_1^*). \tag{32}$$

It follows easily from the way in which a characteristic function is associated to a representation that if two c.n.c. representations are (unitarily)

equivalent then the associated characteristic functions are isomorphic in the sense of Definition 3.28. Conversely, a moment's reflection on Lemma 3.23 and Proposition 3.25 reveals immediately that given two isomorphic characteristic functions, the associated representations are unitarily equivalent. We may therefore summarize our analysis to this point in the following theorem that asserts that the isomorphism class of a characteristic function of a c.n.c. representation is a complete unitary invariant for the representation.

Theorem 3.29 *Two c.n.c. representations are unitarily equivalent if and only if the associated characteristic functions are isomorphic.*

Remark 3.30 *The notion of isomorphism between two characteristic functions Θ and Θ' was defined using the operators $\check{\Theta}$ and $\check{\Theta}'$. One can also write an isomorphism directly in terms of Θ and Θ'. For this, note first that if Hilbert space isomorphisms $W_i : \mathcal{E}_i \to \mathcal{E}_i'$ intertwining τ_i and τ_i', $i = 1, 2$, exist, then $\tau_1 \oplus \tau_2$ and $\tau_1' \oplus \tau_2'$ have the same kernels. So, if we choose a common representation π_0 to define the supplements $\mathcal{E}$ and $\mathcal{E}'$ for these representations, then the W_i's may be extended to a Hilbert space isomorphism $W : \mathcal{E} \to \mathcal{E}'$ that intertwines τ and τ'. On the other hand, if such a W exists, then it restricts to give W_i's that intertwine τ_i and τ_i'. Also, equation (32) is equivalent to the equation*

$$\Theta' \otimes I_{\mathcal{E}'} = C(\Theta \otimes I_{\mathcal{E}})C^*$$

where C is the unitary operator $C = U'(I_{\mathcal{F}(E)} \otimes W)U^* : F(E^\tau) \otimes_\iota \mathcal{E} \to F(E^{\tau'}) \otimes_{\iota'} \mathcal{E}'$ and U and U' are the evident Fourier transforms. In fact, one can show that for $\eta \in (E^{\otimes k})^\tau = (E^\tau)^{\otimes k} \subseteq F(E^\tau)$ and $h \in \mathcal{E}$,

$$C(\eta \otimes h) = (I \otimes W)\eta W^* \otimes Wh,$$

where $(I \otimes W)\eta W^*$ is a map from $\mathcal{E}'$ to $E^{\otimes k} \otimes \mathcal{E}'$ that lies in the τ-dual of $E^{\otimes k}$, which may be identified with $(E^\tau)^{\otimes k}$ by Proposition 2.8. Consequently, the map $X \mapsto X'$, where $X' \otimes I_{\mathcal{E}'} = C(X \otimes I_{\mathcal{E}})C^*$, is an isomorphism of $H^\infty(E^\tau)$ onto $H^\infty(E^{\tau'})$. Once we use this map to identify the two algebras, we see that the two characteristic functions are isomorphic in the sense of Definition 3.28 if they are identified via this map. Since we do not use this remark in the rest of the paper, we shall omit further details.

3.8 Models and Characteristic Functions: Completing the Circle

Lemma 3.31 *Let (T, σ) be a c.n.c. representation of the W^*-correspondence on a Hilbert space, let $\mathcal{D}$ and $\mathcal{D}_*$ be the defect spaces, let $\Theta = \Theta_T$ be its characteristic operator and let $\Delta := \Delta_{\Theta_T} = (I - \Theta^*\Theta)^{1/2}$ Then:*

(i) There is no non zero vector $x \in \mathcal{D}$ such that $x = P_{\mathcal{D}}\Theta^* P_{\mathcal{D}_*}\Theta x$.

(ii) $\overline{\Delta(\mathcal{F}(E) \otimes_{\sigma_1} \mathcal{D})} = \overline{\Delta((\mathcal{F}(E) \otimes_{\sigma_1} \mathcal{D}) \ominus \mathcal{D})}$, where $\sigma_1 = \sigma \circ \varphi$.

Proof. It follows from the proof of Theorem 3.21 (see equation (19)) that $P_{\mathcal{D}_*}\Theta|\mathcal{D} = -\tilde{T}$. So (i) amounts to the fact that the kernel of the positive operator $D = (I - T^*T)^{1/2}$ restricted to the range of D (i.e. to $\mathcal{D}$) is trivial. Since this is obvious, (i) is proved. To prove (ii) note first that $P_\infty(K) = \overline{span}\{V(\xi)P_\infty(k) : \xi \in E, k \in K\} = \overline{span}\{V(\xi)P_\infty(k) : \xi \in E, k \in L_\infty(\mathcal{D})\} = \overline{span}\{P_\infty(V(\xi)k) : \xi \in E, k \in L_\infty(\mathcal{D})\} = \overline{P_\infty(L_\infty(\mathcal{D}) \ominus \mathcal{D})}$. So if $x \in \mathcal{F}(E) \otimes_{\sigma_1} \mathcal{D}$ and if Φ_2 is the isometry defined in Proposition 3.25, then $\Phi_2(\Delta x)$ lies in $P_\infty(K)$. Hence $\Phi_2(\Delta x) = \lim P_\infty y_n$ for some $y_n \in L_\infty(\mathcal{D}) \ominus \mathcal{D}$ and so $\Delta x = \lim \Phi_2^* P_\infty y_n = \lim \Delta_Y(W_{\mathcal{D}}^* y_n)$. It follows that $\Delta x \in \overline{\Delta((\mathcal{F}(E) \otimes_{\sigma_1} \mathcal{D}) \ominus \mathcal{D})}$. $\Box$

Definition 3.32 *Let $\Theta = (\Theta, \mathcal{E}_1, \mathcal{E}_2, \tau_1, \tau_2)$ be a characteristic function and let $\Delta := (I - \check{\Theta}^*\check{\Theta})^{1/2}$.*

(i) We say that Θ is pure *if there is no non-zero vector x in $\mathcal{E}_1$ so that $x = P_{\mathcal{E}_1}\check{\Theta}^* P_{\mathcal{E}_2}\check{\Theta}x$.*

(ii) We say that Θ is predictable *in case*

$$\overline{\Delta(\mathcal{F}(E) \otimes_{\tau_1} \mathcal{E}_1)} = \overline{\Delta((\mathcal{F}(E) \otimes_{\tau_1} \mathcal{E}_1) \ominus \mathcal{E}_1)}.$$

Remark 3.33 *The reason for the term "predictable" derives from the role of Hardy spaces in the setting of prediction theory. Recall that if $M = \mathbb{C} = E$, then the Fock space $\mathcal{F}(E)$ may be identified with the Hardy space $H^2(\mathbb{T})$. So, if $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{C}$ also, then $\mathcal{F}(E) \otimes_{\tau_1} \mathcal{E}_1 = H^2(\mathbb{T})$ as well, and a characteristic function is simply a function $\theta \in H^\infty(\mathbb{T})$ such that $\|\theta\| \leq 1$, i.e., θ is a Schur function. (The function θ is pure if and only if θ is not constant, by the maximum modulus principle.) The function $\delta := (1 - |\theta|^2)^{1/2}$ lies in*

$L^\infty(\mathbb{T})$. *To say that* θ *is predictable is the same thing as saying that* $\overline{\delta H^2(\mathbb{T})} = \overline{\delta H_0^2(\mathbb{T})}$, *where* $H_0^2(\mathbb{T})$ *is the space of those functions in* $H^2(\mathbb{T})$ *that vanish at the origin. The connection with prediction theory is this: Suppose* $\{\xi_n\}_{n\in\mathbb{Z}}$ *is a stationary Gaussian process with covariance matrix* $\{\hat{\delta}(n-m)\}_{n,m\in\mathbb{Z}}$. *Then the future,* $\bigvee_{n>0}\xi_n$, *is contained in the past,* $\bigvee_{n\leq 0}\xi_n$, *i.e., the process* $\{\xi_n\}_{n\in\mathbb{Z}}$ *is predictable, if and only if* $\overline{\delta H^2(\mathbb{T})} = \overline{\delta H_0^2(\mathbb{T})}$. *We note in passing that* θ *is predictable if and only if* $\overline{\delta H^2(\mathbb{T})} = L^2(\mathbb{T})$ *and that this is also equivalent to the assertion that* $\ln(\delta) \notin L^1(\mathbb{T})$ *by Szegö's theorem.*

Remark 3.34 *Let* Θ *be a characteristic function. Note that, for all* ξ, ζ *in* $E^{\otimes n}$, $\check{\Theta}$ *commutes with both* $T_\xi \otimes I_{\mathcal{E}_1}$ *and* $T_\zeta^* T_\xi \otimes I_{\mathcal{E}_1}$, *since* $T_\zeta^* T_\xi \in \varphi_\infty(M)$. *Thus* $(T_\zeta^* \otimes I)\check{\Theta}(T_\xi \otimes I) = (T_\zeta T_\xi^* \otimes I)\check{\Theta} = \check{\Theta}(T_\zeta^* \otimes I)(T_\xi \otimes I)$. *It follows that* $(T_\zeta^* \otimes I)\check{\Theta}$ *and* $\check{\Theta}(T_\zeta^* \otimes I)$ *are equal when restricted to* $E^{\otimes m} \otimes \mathcal{E}_1$ *for* $m \geq n$.

Lemma 3.35 *Let* $\Theta = (\Theta, \mathcal{E}_1, \mathcal{E}_2, \tau_1, \tau_2)$ *be a characteristic function that is pure and predictable. Form its canonical model* $(T, \sigma) := (T_\Theta, \sigma_\Theta)$ *on the Hilbert space* $H(\Theta)$ *and the isometric representation* $(V, \rho) := (V_\Theta, \rho_\Theta)$ *on the Hilbert space* $K(\Theta)$ *as described in Lemma 3.23. Then* (V, ρ) *is* minimal *as an isometric dilation of* (T, σ).

Proof. We already know that (V, ρ) is an isometric dilation of (T, σ) by definition. So we need only prove minimality. For this, write $\mathcal{K}$ for the subspace

$$\mathcal{K} = \overline{span}\{V(\xi)H(\Theta) : \xi \in E\}.$$

We shall show that $\mathcal{K} = K(\Theta)$. Fix a vector $x \in K(\Theta) \ominus \mathcal{K}$. Since x is orthogonal to $H(\Theta)$, we can write $x = \check{\Theta}w_0 + \Delta w_0$ for some $w_0 \in \mathcal{F}(E) \otimes_{\tau_1} \mathcal{E}_1$, where as usual $\Delta := (I - \check{\Theta}^*\check{\Theta})^{1/2}$. For every $n \geq 1$ and every $\xi \in E^{\otimes n}$, $V(\xi)^* x \in H(\Theta)^\perp$ and we can find $w(\xi) \in \mathcal{F}(E) \otimes \mathcal{E}_1$ such that

$$V(\xi)^*(\check{\Theta}w_0 + \Delta w_0) = \check{\Theta}w(\xi) + \Delta w(\xi).$$

We now write S for the operator S_Θ in Lemma 3.23 and conclude from the previous equation that $(T_\xi^* \otimes I)\check{\Theta}w_0 = \check{\Theta}w(\xi)$ and $S(\xi)^*\Delta w_0 = \Delta w(\xi)$. Hence, for every ξ, ζ in $E^{\otimes n}$ we have

$$\check{\Theta}^*(T_\zeta T_\xi^* \otimes I)\check{\Theta}w_0 = \check{\Theta}^*\check{\Theta}(T_\zeta \otimes I)w(\xi)$$

and

$$\Delta S(\zeta)S(\xi)^*\Delta w_0 = \Delta^2(T_\zeta \otimes I)w(\xi),$$

where we used the facts that $\check{\Theta}$ commutes with $T_\zeta \otimes I$ and that, by definition, $S(\zeta)\Delta = \Delta(T_\zeta \otimes I))$. Adding these two equations gives

$$\check{\Theta}^*(T_\zeta T_\xi^* \otimes I)\check{\Theta} w_0 + \Delta S(\zeta)S(\xi)^*\Delta w_0 = (T_\zeta \otimes I)w(\xi). \tag{33}$$

We shall write e_i (respectively, f_i) for the projection of $\mathcal{F}(E) \otimes_{\tau_1} \mathcal{E}_1$ (respectively, $\mathcal{F}(E) \otimes_{\tau_2} \mathcal{E}_2$) onto $E^{\otimes i} \otimes_{\tau_1} \mathcal{E}_1$ (respectively, $E^{\otimes i} \otimes_{\tau_2} \mathcal{E}_2$). Note that, for $\zeta \in E^{\otimes n}$ as above, we have $e_i(T_\zeta \otimes I)w(\xi) = 0$ if $i < n$. Thus, for $i < n$,

$$e_i(\check{\Theta}^*(T_\zeta T_\xi^* \otimes I)\check{\Theta} w_0 + \Delta S(\zeta)S(\xi)^*\Delta w_0) = 0.$$

It will be convenient to write (R, ϕ) for the (isometric) representation of E on $\mathcal{F}(E) \otimes_{\tau_2} \mathcal{E}_2$ defined by $R(\xi) = T_\xi \otimes I_{\mathcal{E}_2}$ (for $\xi \in E$) and $\phi(a) = \varphi_\infty(a) \otimes I$ for $a \in M$. Then the maps $\tilde{R}_n : E^{\otimes n} \otimes_{\tau_2 \circ \varphi_\infty} \mathcal{F}(E) \otimes_{\tau_2} \mathcal{E}_2 \to \mathcal{F}(E) \otimes_{\tau_2} \mathcal{E}_2$ are defined in the usual way. For $\zeta, \xi \in E^{\otimes n}$ we write $\zeta \otimes \xi^*$ for the operator $\zeta \otimes \xi^*$ on $E^{\otimes n}$ defined by the formula $(\zeta \otimes \xi^*)\xi' = \zeta\langle \xi, \xi' \rangle$. The C^*-algebra generated by these operators is written $K(E^{\otimes n})$ and it is σ-weakly dense in the W^*-algebra $\mathcal{L}(E^{\otimes n})$. We have $S(\zeta)S(\xi)^* = \tilde{S}_n((\zeta \otimes \xi^*) \otimes I)\tilde{S}_n^*$ and $T_\zeta T_\xi^* \otimes I = \tilde{R}_n((\zeta \otimes \xi^*) \otimes I)\tilde{R}_n^*$. Hence, for every $K \in K(E^{\otimes n})$ and every $i < n$,

$$e_i(\check{\Theta}^* \tilde{R}_n(K \otimes I_{\mathcal{F}(E)\otimes\mathcal{E}_2})\tilde{R}_n^* \check{\Theta} w_0 + \Delta \tilde{S}_n(K \otimes I_{\Delta(\mathcal{F}(E)\otimes\mathcal{E}_1)})\tilde{S}_n^* \Delta w_0) = 0.$$

Noting that $I_{E^{\otimes n}}$ is in the σ-weak closure of $K(E^{\otimes n})$ we conclude that

$$e_i(\check{\Theta}^* \tilde{R}_n \tilde{R}_n^* \check{\Theta} w_0 + \Delta \tilde{S}_n \tilde{S}_n^* \Delta w_0) = 0$$

for $i < n$. But $\tilde{R}_n \tilde{R}_n^* = \sum_{j=n}^{\infty} f_j$, on the one hand, and $\tilde{S}_n \tilde{S}_n^* = I$ by our assumption that Θ is predictable. Thus $e_i(\check{\Theta}^*(\sum_{j=n}^{\infty} f_j)\check{\Theta} w_0 + \Delta^2 w_0) = 0$ and, since $\Delta^2 = I - \check{\Theta}^*(\sum_{j=0}^{\infty} f_j)\check{\Theta}$, we have $e_i(w_0 - \check{\Theta}^*(\sum_{j=0}^{n-1} f_j)\check{\Theta} w_0) = 0$. But also $(\sum_{j=0}^{n-1} f_j)\check{\Theta} w_0 = (\sum_{j=0}^{n-1} f_j)\check{\Theta}(\sum_{k=0}^{n-1} e_k)w_0$ and we get the following equation, for every $i < n$,

$$e_i w_0 - e_i \check{\Theta}^*(\sum_{j=0}^{n-1} f_j)\check{\Theta}(\sum_{k=0}^{n-1} e_k)w_0 = 0. \tag{34}$$

Setting $n = 1$ and $i = 0$ we obtain in particular the equation $e_0 w_0 = e_0 \check{\Theta}^* f_0 \check{\Theta} e_0 w_0$. Since Θ is assumed to be pure, $e_0 w_0 = 0$. Now set $n = 2$

and $i = 1$ in equation (34) and use the fact that $f_0\check{\Theta}w_0 = f_0\check{\Theta}e_0 = 0$ to conclude that

$$e_1w_0 = e_1\check{\Theta}^*f_1\check{\Theta}e_1w_0. \tag{35}$$

In order to "bootstrap" purity to this equation we first fix $\zeta \in E$ and, using Remark 3.34, we compute

$$(T_\zeta^* \otimes I)e_1w_0 = (T_\zeta^* \otimes I)e_1\check{\Theta}^*f_1\check{\Theta}e_1w_0 = e_0\check{\Theta}^*f_0(T_\zeta^* \otimes I)\check{\Theta}e_1w_0 =$$

$$= e_0\check{\Theta}^*f_0\check{\Theta}(T_\zeta^* \otimes I)e_1w_0.$$

Now we can appeal to the purity of Θ to conclude that $(T_\zeta^* \otimes I)e_1w_0 = 0$. Since this holds for all $\zeta \in E$, $e_1w_0 = 0$. Continuing in this way we see that $e_nw_0 = 0$ for all $n \geq 0$. Thus $w_0 = 0$ and, consequently, $x = 0$. $\Box$

Lemma 3.36 *Let Θ be a characteristic function that is pure and predictable and adopt the notation from Lemma 3.23. For $i \geq 1$ set*

$$\mathcal{K}_i := \overline{span}\{V_\Theta(\xi)h \mid \xi \in E^{\otimes i},\ h \in H(\Theta)\ \}$$

and for $j \geq 0$ set

$$\mathcal{M}_j := \{\check{\Theta}x + \Delta_{\check{\Theta}}x : x \in E^{\otimes j} \otimes \mathcal{E}_1\},$$

where, for $j = 0$, $E^{\otimes 0} \otimes \mathcal{E}_1$ is $\mathcal{E}_1$. Then,

$$\mathcal{M}_0 = (I_{K(\Theta)} - P_\Theta)(\mathcal{K}_1).$$

Proof. As usual, write Δ for $(I - \check{\Theta}^*\check{\Theta})^{1/2}$. First we note that the map taking $x \in \mathcal{F}(E)\otimes\mathcal{E}_1$ to $\check{\Theta}x+\Delta x \in K(\Theta)$ is an isometry, since $\check{\Theta}^*\check{\Theta}+\Delta^2 = I$, and, consequently, that for $i \neq j$, $\mathcal{M}_i$ is orthogonal to $\mathcal{M}_j$. Also, we note that for $x \in E^{\otimes j}\otimes\mathcal{E}_1$ and $\xi \in E$, $V_\Theta(\xi)(\check{\Theta}x+\Delta x) = (T_\xi\otimes I)\check{\Theta}x+S_\Theta(\xi)\Delta x = \check{\Theta}(T_\xi \otimes I)x + \Delta(T_\xi \otimes I)x$. Hence $V_\Theta(E)\mathcal{M}_j \subseteq \mathcal{M}_{j+1}$, where we abbreviate $\overline{span}\{V_\Theta(\xi)x \mid \xi \in E,\ x \in \mathcal{M}_j\}$ by $V_\Theta(E)\mathcal{M}_j$. It is also clear that $V_\Theta(E)\mathcal{K}_i \subseteq \mathcal{K}_{i+1}$.

Next we show that for $j \geq 1$, $\mathcal{K}_1$ is orthogonal to $\mathcal{M}_j$. Indeed, let $j \geq 1$, let $\zeta \in E$, let $\theta \in E^{\otimes(j-1)}$ and let $h \in \mathcal{E}_1$. Then, for $\xi \in E$, we have $V(\xi)^*(\check{\Theta}(\zeta\otimes\theta\otimes h)+\Delta(\zeta\otimes\theta\otimes h)) = (T_\xi^*\otimes I)\check{\Theta}(\zeta\otimes\theta\otimes h)+S_\Theta(\xi)^*\Delta(\zeta\otimes\theta\otimes h)$. Using Remark 3.34 and the fact that $\Delta(\zeta \otimes \theta \otimes h) = \Delta(T_\zeta \otimes I)(\theta \otimes h) = S_\Theta(\zeta)\Delta(\theta\otimes h)$ we find that $V(\xi)^*(\check{\Theta}(\zeta\otimes\theta\otimes h)+\Delta(\zeta\otimes\theta\otimes h)) = \check{\Theta}(T_\xi^*\otimes I)(\zeta\otimes \theta \otimes h) + S_\Theta(\xi)^*S_\Theta(\zeta)\Delta(\theta \otimes h) = \check{\Theta}(\langle\xi,\zeta\rangle\theta \otimes h) + \Delta(\langle\xi,\zeta\rangle\theta \otimes h) \in H(\Theta)^\perp$.

It follows that $\mathcal{K}_1$ is orthogonal to $\mathcal{M}_j$, $j \geq 1$. Since $\mathcal{M}_j = (I - P_\Theta)(\mathcal{M}_j)$, we conclude that $(I - P_\Theta)\mathcal{K}_1$ is orthogonal to $\mathcal{M}_j$ for all $j \geq 1$. But it is also orthogonal to $H(\Theta)$ and we have $K(\Theta) = H(\Theta) \oplus \sum_{j=0}^{\infty} \oplus \mathcal{M}_j$. Thus

$$(I - P_\Theta)(\mathcal{K}_1) \subseteq \mathcal{M}_0. \tag{36}$$

¿From (36) it follows that $\mathcal{K}_1 \subseteq \mathcal{M}_0 \oplus H(\Theta)$. Applying $V_\Theta(E)$ to this we find that $\mathcal{K}_2 \subseteq \mathcal{M}_1 \oplus \mathcal{K}_1$. A second application of $V_\Theta(E)$ yields $\mathcal{K}_3 \subseteq (\mathcal{M}_2 \oplus \mathcal{M}_1) + \mathcal{K}_1$. Continuing by induction we find that for every $i \geq 2$,

$$\mathcal{K}_i \subseteq \mathcal{K}_1 + \sum_{j=1}^{i-1} \oplus \mathcal{M}_j. \tag{37}$$

Now suppose $y \in \mathcal{M}_0 \ominus (I - P_\Theta)(\mathcal{K}_1)$. Then $y = (I - P_\Theta)y \in \mathcal{K}_1^\perp$. Since $y \in \mathcal{M}_0$, y is also orthogonal to $\mathcal{M}_j$ for every $j \geq 1$. By (37), y is orthogonal to $\mathcal{K}_i$ for every $i \geq 1$. But $y \in H(\Theta)^\perp$ and, by the minimality of (V_Θ, ρ_Θ), $H(\Theta) + \sum \mathcal{K}_i$ is dense in $K(\Theta)$. Thus $y = 0$ and this, combined with the inclusion (36)) completes the proof. $\square$

Lemma 3.37 *Let $(\Theta, \mathcal{E}_1, \mathcal{E}_2, \tau_1, \tau_2)$ be a pure and predictable characteristic function, let $(T, \sigma) = (T_\Theta, \sigma_\Theta)$ be its canonical model acting on $H = H(\Theta)$, and let $\mathcal{D}$ and $\mathcal{D}_*$ be the defect spaces associated with (T, σ). Then:*

(i) The spaces $\mathcal{E}_1$ and $\mathcal{D}$ are isomorphic as left M-modules;i.e. there is a unitary operator $W_1 : \mathcal{E}_1 \to \mathcal{D}$ such that, for every $a \in M$,

$$W_1 \tau_1(a) = (\varphi(a) \otimes I_H) W_1.$$

(ii) The spaces $\mathcal{E}_2$ and $\mathcal{D}_*$ are isomorphic as left M-modules ; i.e. there is a unitary operator $W_2 : \mathcal{E}_2 \to \mathcal{D}_*$ such that, for every $a \in M$,

$$W_2 \tau_2(a) = \sigma(a) W_2.$$

Proof. Write (V, ρ) for the minimal isometric dilation of (T, σ) as constructed in (7) and the discussion preceding it. The representation space of (V, ρ) is $K = H \oplus (\mathcal{F}(E) \otimes_{\sigma_1} \mathcal{D})$. From the uniqueness of the minimal isometric dilation [22, Proposition 3.2] and Lemma 3.35, it follows that there is a unitary operator $W : K(\Theta) \to K$ such that W maps $H(\Theta)$ onto H and satisfies the equations $V(\xi)W = WV_\Theta(\xi)$, $\xi \in E$, and $\rho(a)W = W\rho_\Theta(a)$, $a \in$

M. Write $W_1 h = W(\check{\Theta}h + \Delta h)$ for $h \in \mathcal{E}_1$, where $\Delta := (I - \check{\Theta}^*\check{\Theta})^{1/2}$. Then, in the notation of Lemma 3.36, $W_1(\mathcal{E}_1) = W\mathcal{M}_0 = W(I - P_{H(\Theta)})\mathcal{K}_1 = W(I - P_{H(\Theta)})V_\Theta(E)H(\Theta) = (I - P_{H(\Theta)})WV_\Theta(E)W^*WH(\Theta) = (I - P_{H(\Theta)})V(E)H = \mathcal{D}$, where the last equality follows from equation (7). Recall that the map $x \mapsto \check{\Theta}x + \Delta x$ is an isometry defined on $\mathcal{F}(E) \otimes \mathcal{E}_1$. Hence W_1 is indeed a unitary operator from $\mathcal{E}_1$ onto $\mathcal{D}$. Now fix $a \in M$ and $h \in \mathcal{E}_1$ and recall that $\mathcal{D} \subseteq E \otimes H$ and $\rho(a)|\mathcal{D} = (\varphi(a) \otimes I_{H(S)})|\mathcal{D}$. We have

$$\begin{aligned}&(\varphi(a) \otimes I_H)W_1 h = \rho(a)W(\check{\Theta}h + \Delta h) = W\rho_\Theta(a)(\check{\Theta}h + \Delta h)\\ &= W((\varphi_\infty(a) \otimes I)\check{\Theta}h + \Delta\tau_1(a)h) = W(\check{\Theta}\tau_1(a)h + \Delta\tau_1(a)h) = W_1\tau_1(a)h.\end{aligned}$$

This proves (i).

To prove the other assertion, recall first from Lemma 3.4 that K_0 is the range of the projection $I - \tilde{V}\tilde{V}^*$ (in fact, we can write $K_0 = K \ominus V(E)K$) and there is an isometry u from K_0 onto $\mathcal{D}_*$. Note that we may view $\mathcal{E}_2$ as the first summand of $\mathcal{F}(E) \otimes \mathcal{E}_2$ and that when we do, we can write $\mathcal{E}_2 = (\mathcal{F}(E) \otimes \mathcal{E}_2) \ominus \overline{span}\{(T_\xi \otimes I)(\mathcal{F}(E) \otimes \mathcal{E}_2) \mid \xi \in E\}$. Since $S_\Theta(E)\Delta(\mathcal{F}(E) \otimes \mathcal{E}_1) = \Delta((\mathcal{F}(E) \otimes \mathcal{E}_1) \ominus \mathcal{E}_1) = \Delta(\mathcal{F}(E) \otimes \mathcal{E}_1)$, we have $\mathcal{E}_2 = K(\Theta) \ominus V_\Theta(E)K(\Theta) = W^*K \ominus W^*V(E)WW^*K = W^*(K \ominus V(E)K) = W^*K_0$. Thus, setting $W_2 = uW|\mathcal{E}_2$, we obtain a unitary operator from $\mathcal{E}_2$ onto $\mathcal{D}_*$. Finally, for $a \in M$ and $h \in \mathcal{E}_2 \subseteq K(\Theta)$,

$$W_2\tau_2(a)h = uW\rho_\Theta(a)h = u\rho(a)Wh = \sigma(a)W_2 h$$

where the last equality follows from Lemma 3.4 (iii). $\square$

Theorem 3.38 *Let $(\Theta, \mathcal{E}_1, \mathcal{E}_2, \tau_1, \tau_2)$ be a pure and predictable characteristic function and let $(T, \sigma) = (T_\Theta, \sigma_\Theta)$ on $H := H(\Theta)$ be the associated canonical model. Then this representation is c.n.c and its characteristic function $(\hat{\Theta}_T, \mathcal{D}, \mathcal{D}_*, (\varphi \otimes I_H)|\mathcal{D}, \sigma|\mathcal{D}_*)$ is isomorphic to $(\Theta, \mathcal{E}_1, \mathcal{E}_2, \tau_1, \tau_2)$.*

Proof. We continue with the notation of the proof of Lemma 3.37. In particular, W_1 will denote the Hilbert space isomorphism from $\mathcal{E}_1$ to $\mathcal{D}$ constructed there, while W_2 will denote the Hilbert space isomorphism from $\mathcal{E}_2$ to $\mathcal{D}_*$. Also, W will be the unitary operator from $K(\Theta)$ onto K, where K is the space of the minimal isometric dilation (V, ρ) of (T, σ) as in the proof of Lemma 3.37. It is shown there that W maps $\mathcal{E}_2$ onto K_0 and it intertwines V_Θ and V. Thus it maps $\overline{\mathcal{F}(E) \otimes \mathcal{E}_2}$ onto $Q_\infty(K)$. Since $W(H(\Theta)) = H$, $H \cap P_\infty(K) = W(H(\Theta) \cap \overline{\Delta(\mathcal{F}(E) \otimes \mathcal{E}_1)})$. But if $y \in H(\Theta) \cap \overline{\Delta(\mathcal{F}(E) \otimes \mathcal{E}_1)}$

then, for every $x \in \mathcal{F}(E) \otimes \mathcal{E}_1$, y is orthogonal to $\check{\Theta}x + \Delta x$ and also y is orthogonal to $\check{\Theta}x \in \mathcal{F}(E) \otimes \mathcal{E}_2$. Thus y is orthogonal to Δx for every such x and it follows that $y = 0$. Hence $H \cap P_\infty(K) = \{0\}$ and, consequently, (T, σ) is a c.n.c. representation.

Since W maps $\mathcal{F}(E) \otimes \mathcal{E}_2$ onto $Q_\infty(K)$, it follows that $Q_\infty W \Delta x = 0$ for $x \in \mathcal{F}(E) \otimes \mathcal{E}_1$. Also, recall that $W_2 = uW|\mathcal{E}_2$ and, for $\xi \in \mathcal{F}(E)$ and $h \in \mathcal{E}_2$ we have $(I_{\mathcal{F}(E)} \otimes W_2)(\xi \otimes h) = \xi \otimes uWh = (I \otimes u)(\xi \otimes Wh) = (I \otimes u)W_\infty^* Q_\infty V(\xi)Wh = (I \otimes u)W_\infty^* Q_\infty W(\xi \otimes h)$. Thus

$$I_{\mathcal{F}(E)} \otimes W_2 = (I_{\mathcal{F}(E)} \otimes u)W_\infty^* Q_\infty W.$$

So from the definition of Θ_T, Definition 3.10, we find that for every $h \in \mathcal{E}_1$,

$$\begin{aligned} \Theta_T W_1 h &= \Theta_T W(\check{\Theta}h + \Delta h) = \Theta_T W \check{\Theta} h \\ &= (I_{\mathcal{F}(E)} \otimes u)W_\infty^* Q_\infty W \check{\Theta} h = (I_{\mathcal{F}(E)} \otimes W_2)\check{\Theta} h. \end{aligned}$$

Hence, for $\xi \otimes d \in \mathcal{F}(E) \otimes \mathcal{D}$ and $h := W_1^* d \in \mathcal{E}_1$, we have $(I_{\mathcal{F}(E)} \otimes W_2)\check{\Theta}(I_{\mathcal{F}(E)} \otimes W_1^*)(\xi \otimes d) = (I_{\mathcal{F}(E)} \otimes W_2)\check{\Theta}(\xi \otimes h) = (I_{\mathcal{F}(E)} \otimes W_2)(T_\xi \otimes I_{\mathcal{E}_2})\check{\Theta} h = (T_\xi \otimes I_{\mathcal{D}_*})(I_{\mathcal{F}(E)} \otimes W_2)\check{\Theta} h = (T_\xi \otimes I_{\mathcal{D}_*})\Theta_T W_1 h = \Theta_T (T_\xi \otimes I_{\mathcal{D}})d = \Theta_T(\xi \otimes d)$. Therefore

$$(I_{\mathcal{F}(E)} \otimes W_2)\check{\Theta}(I_{\mathcal{F}(E)} \otimes W_1^*) = \Theta_T,$$

as was to be proved. $\square$

4 Commutants of Models

In [22, Theorem 4.4] we proved a commutant lifting theorem for completely contractive representations of tensor algebras. The analysis there extends without difficulty to σ-weakly continuous representations of Hardy algebras. However, with the analysis in [28] available to us and the results of the preceding section, it is possible to give a refined version of the commutant lifting theorem, at least in the context of $C_{\cdot 0}$ representations. The theorem we shall prove in this section generalizes Theorem 6.1 of [32].

First recall that if (T, σ) is a $C_{\cdot 0}$ representation of E on a Hilbert space H, if $\Theta = \hat{\Theta}_T$ is the characteristic function associated to the characteristic operator $(\Theta_T, \mathcal{D}, \mathcal{D}_*, \tau_1, \tau_2)$, and if $(T_\Theta, \sigma_\Theta)$ is the canonical model built from Θ, then the Hilbert space of the minimal isometric dilation of $(T_\Theta, \sigma_\Theta)$, $K(\Theta)$, is $\mathcal{F}(E) \otimes_{\tau_2} \mathcal{D}_*$, by virtue of Theorems 3.19 and 3.25. (A bit more

completely, Theorem 3.19 guarantees that $\hat{\Theta}_T$ is inner if (T, σ) is $C_{\cdot 0}$. Also, Lemma 3.7 guarantees that the minimal isometric dilation of (T, σ) is an induced representation if (and only if) (T, σ) is $C_{\cdot 0}$. And, Theorem 3.25 identifies the form of that induced representation.) The model space $H(\Theta)$ is $(\mathcal{F}(E) \otimes_{\tau_2} \mathcal{D}_*) \ominus \Theta_T(\mathcal{F}(E) \otimes_{\tau_1} \mathcal{D})$ in this case. Recall, too, that $(\mathcal{G}, \tau)$ is a fixed supplement of τ_1 and τ_2 and that $\mathcal{F}(E) \otimes_\tau \mathcal{G}$ decomposes as $\mathcal{F}(E) \otimes_\tau \mathcal{G} = (\mathcal{F}(E) \otimes_{\pi_0} H_0) \oplus (\mathcal{F}(E) \otimes_{\tau_1} \mathcal{D}) \oplus (\mathcal{F}(E) \otimes_{\tau_2} \mathcal{D}_*)$ (equation (15)). A moment's reflection reveals that if v_2 is the isometric embedding of $\mathcal{D}_*$ in $\mathcal{G}$ that sends d_* in $\mathcal{D}_*$ to $(0, 0, d_*)^{tr}$, then $I \otimes v_2$ is an isometric embedding of $\mathcal{F}(E) \otimes_{\tau_2} \mathcal{D}_*$ in $\mathcal{F}(E) \otimes_\tau \mathcal{G}$ that intertwines the two induced representations of $H^\infty(E)$ and that maps $H(\Theta)$ onto the space

$$(\mathcal{F}(E) \otimes_\tau \mathcal{G}) \ominus \Theta_T(\mathcal{F}(E) \otimes_\tau \mathcal{G}),$$

where here Θ_T is treated as the matrix in equation (16). On the other hand, the canonical equivalence Φ from $K(\Theta) = \mathcal{F}(E) \otimes_{\tau_2} \mathcal{D}_*$ to the Hilbert space K of the minimal isometric dilation (V, ρ) of (T, σ) is a Hilbert space isomorphism that intertwines $V \times \rho$ and the induced representation $\tau_2^{\mathcal{F}(E)}$, maps $H(\Theta)$ onto H and implements a unitary equivalence between (T, σ) and $(T_\Theta, \sigma_\Theta)$ (see Theorem 3.25). Hence, if $U : \mathcal{F}(E) \otimes_\tau \mathcal{G} \to \mathcal{F}(E^\tau) \otimes_\iota \mathcal{G}$ is the Fourier transform from Remark 2.11, and if U_0 is the composition $U_0 := U(I \otimes v_2)(\Phi^{-1}|H)$, i.e., if U_0 is built from the following diagram

$$H \subseteq K \xrightarrow{\Phi^{-1}} K(\Theta) \xrightarrow{I \otimes v_2} \mathcal{F}(E) \otimes_\tau \mathcal{G} \xrightarrow{U} \mathcal{F}(E^\tau) \otimes_\iota \mathcal{G},$$

then U_0 is an isometry mapping H into $\mathcal{F}(E^\tau) \otimes_\iota \mathcal{G}$ and has the property that for every $\Xi \in H^\infty(E^\tau)$, $U_0^*(\Xi \otimes I_{\mathcal{G}})U_0$ commutes with $T \times \sigma(H^\infty(E))$.

Theorem 4.1 *Let π be a completely contractive σ-weakly continuous representation of $H^\infty(E)$ on the Hilbert space H such that the associated covariant representation of E, (T, σ), is a $C_{\cdot 0}$-representation. Let $U_0 : H \to \mathcal{F}(E^\tau) \otimes_\iota \mathcal{G}$ be the isometric embedding just described. Then for every $X \in B(H)$ that commutes with $\pi(H^\infty(E))$, there is an $\Xi \in H^\infty(E^\tau)$ such that*

(i) $\|\Xi\| = \|X\|$, *and*

(ii) $X = U_0^*(\Xi \otimes I_{\mathcal{G}})U_0$.

Proof. We have already noted that every X of the form in (ii) commutes with $\pi(H^\infty(E))$ and of course $\|X\| \leq \|\Xi \otimes I_{\mathcal{E}}\| = \|\Xi\|$ since $\iota^{\mathcal{F}(E^\tau)}$ is faithful by Remark 2.7. But the converse results from [22, Theorem 4.4] as follows. Given $X \in B(H)$ that commutes with $\pi(H^\infty(E))$, Theorem 4.4 of [22] produces an operator Y on the Hilbert space K of the minimal isometric dilation (V, ρ) of (T, σ) that commutes with (V, ρ), satisfies the equation $\|Y\| = \|X\|$ and satisfies the equation $X = P_H Y|H$. Since (T, σ) is $C_{\cdot 0}$, Lemma 3.7 implies that (V, ρ) is an induced representation. Theorem 3.25 identifies the structure of that induced representation and shows that Φ implements an equivalence between (V, ρ) and the (covariant) representation $\tau_2^{\mathcal{F}(E)}$. The map $I \otimes v_2$ embeds $\mathcal{F}(E) \otimes_{\tau_2} \mathcal{D}_*$ into $\mathcal{F}(E) \otimes_\tau \mathcal{G}$ in such a way that $(I \otimes v_2)\Phi^{-1}(Y)\Phi(I \otimes v_2)^*$ commutes with $\tau^{\mathcal{F}(E)}(H^\infty(E))$. So, since U is the Fourier transform from $\mathcal{F}(E) \otimes_\tau \mathcal{G}$ to $\mathcal{F}(E^\tau) \otimes_\iota \mathcal{G}$, Theorem 2.10 guarantees that $U(I \otimes v_2)\Phi^{-1}(Y)\Phi(I \otimes v_2)^* U^*$ is an operator on $\mathcal{F}(E^\tau) \otimes_\iota \mathcal{G}$ that lies in $\iota^{\mathcal{F}(E^\tau)}(H^\infty(E^\tau))$, i.e., $U(I \otimes v_2)\Phi^{-1}(Y)\Phi(I \otimes v_2)^* U^* = \Xi \otimes I_{\mathcal{G}}$ for a $\Xi \in H^\infty(E^\tau)$. Hence, as a calculation reveals, $U_0^*(\Xi \otimes I_{\mathcal{G}})U_0 = X$ and $\|\Xi\| \leq \|Y\| = \|X\|$. $\square$

Remark 4.2 *If $M = \mathbb{C} = E$, and if (T, σ) is a $C_{\cdot 0}$ representation with 1-dimensional defect spaces, then Theorem 4.1 gives Sarason's original commutant lifting theorem [39].*

5 Invariant Subspaces

In the theory of models for single operators, invariant subspaces are determined by factorizations of the characteristic operator functions. The same is true in our setting. To keep the presentation as simple as possible, we shall restrict our attention to $C_{\cdot 0}$ representations. We shall need to consider factorizations, i.e., compositions, $\Theta = \Theta_1\Theta_2$, where Θ is the necessarily inner characteristic function associated with a $C_{\cdot 0}$-representation and where each Θ_i, $i = 1, 2$, is an inner characteristic function that is *not* necessarily purely contractive. Two such compositions $\Theta = \Theta_1\Theta_2 = \Theta_1'\Theta_2'$ are said to be *equivalent* if $\Theta_1' = \Theta_1(I \otimes V_0)$ and $\Theta_2' = (I \otimes V_0^*)\Theta_2$ for a suitable unitary operator V_0.

Theorem 5.1 *Let (T, σ) a $C_{\cdot 0}$-representation of E on H, with $T \times \sigma$ denoting the associated representation of $H^\infty(E)$, and let $\Theta := \hat{\Theta}_T$ be the inner*

characteristic function of this representation. Then there is a bijection between the subspaces of H that are invariant under $(T \times \sigma)(H^\infty(E))$ and equivalence classes of factorizations $\Theta = \Theta_1\Theta_2$ of Θ as a composition of two inner characteristic functions.

Proof. By Theorem 3.25, we may assume that (T, σ) is $(T_\Theta, \sigma_\Theta)$ for the inner characteristic function $(\Theta, \mathcal{D}, \mathcal{D}_*, \tau_1, \tau_2)$. Hence, the space H is $H(\Theta) = (\mathcal{F}(E) \otimes \mathcal{D}_*) \ominus \Theta(\mathcal{F}(E) \otimes \mathcal{D})$.

Fix a subspace $\mathcal{M} \subseteq H(\Theta)$ that is invariant under $(T \times \sigma)(H^\infty(E))$; that is, for every $\xi \in E$ and $a \in M$, $T_\Theta(\xi)\mathcal{M} \subseteq \mathcal{M}$ and $\sigma_\Theta(a)\mathcal{M} \subseteq \mathcal{M}$. Write $\mathcal{N} = \mathcal{M} \oplus \Theta(\mathcal{F}(E) \otimes \mathcal{D}) \subseteq K(\Theta)$. Recall that $T_\Theta(\xi)$ (for $\xi \in E$) and $\sigma_\Theta(a)$ (for $a \in M$) are the compressions of $T_\xi \otimes I_{\mathcal{D}_*}$ and $\varphi_\infty(a) \otimes I_{\mathcal{D}_*}$, respectively, to $H(\Theta)$. Also recall that $T_\xi \otimes I_{\mathcal{D}_*}$ and $\varphi_\infty(a) \otimes I_{\mathcal{D}_*}$ leave $\Theta(\mathcal{F}(E) \otimes \mathcal{D})$ invariant. It follows that $\mathcal{N}$ is invariant under these operators. Thus, defining $S(\xi)$ and $\pi(a)$ (for $\xi \in E$ and $a \in M$) to be the restrictions of $T_\xi \otimes I_{\mathcal{D}_*}$ and $\varphi_\infty(a) \otimes I_{\mathcal{D}_*}$, respectively, to $\mathcal{N}$, we get an isometric representation of E on $\mathcal{N}$. Since this is the restriction of a pure representation in the sense of [24], meaning that condition (ii) of Lemma 3.7 is satisfied, it is also pure. It follows from the equivalence of (ii) and (iv) in Lemma 3.7 (S, π) is induced. That is, there is a representation ρ of M on a Hilbert space H_0 such that (S, π) is unitarily equivalent to the induced representation on $\mathcal{F}(E) \otimes_\rho H_0$. Hence, there is a unitary operator Θ_1 from $\mathcal{F}(E) \otimes_\rho H_0$ onto $\mathcal{N}$ intertwining the induced representation and (S, π). It is then easy to check that $(\Theta_1, H_0, \mathcal{D}_*, \rho, \tau_2)$ is an inner characteristic function. (Recall that it is not assumed to be purely contractive).

We now write $\Theta_2 = \Theta_1^*\Theta : \mathcal{F}(E) \otimes_{\tau_1} \mathcal{D} \to \mathcal{F}(E) \otimes_\rho H_0$. Clearly, Θ_2 is an isometry (note that the range of Θ is contained in the range of Θ_1) and since Θ_2 evidently intertwines $\rho^{\mathcal{F}(E)}$ and $\tau_1^{\mathcal{F}(E)}$, we see that $(\Theta_2, \mathcal{D}, H_0, \tau_1, \rho)$ is an inner characteristic function (where, again, we do not assume that it is purely contractive).We have $\Theta = \Theta_1\Theta_2$.

So far, starting with an invariant subspace $\mathcal{M}$ of $H(\Theta)$, we obtained a factorization of Θ. Note also that

$$\mathcal{F}(E) \otimes_{\tau_2} \mathcal{D}_* \ominus \Theta_1(\mathcal{F}(E) \otimes_\rho H_0) = H(\Theta) \ominus \mathcal{M}. \tag{38}$$

Now assume that $(\Theta_1, H_0, \mathcal{D}_*, \rho, \tau_2)$ and $(\Theta_2, \mathcal{D}, H_0, \tau_1, \rho)$ are two characteristic functions (not necessarily purely contractive) such that $\Theta = \Theta_1\Theta_2$. Clearly $\Theta(\mathcal{F}(E) \otimes_{\tau_1} \mathcal{D}) \subseteq \Theta_1(\mathcal{F}(E) \otimes_\rho H_0)$. Set

$$\mathcal{M} = \Theta_1(\mathcal{F}(E) \otimes_\rho H_0) \ominus \Theta(\mathcal{F}(E) \otimes_{\tau_1} \mathcal{D}).$$

Then $\mathcal{M} \subseteq H(\Theta)$. Since $\mathcal{M}$ is clearly invariant for $\sigma_\Theta(M)$, we need to show that it is invariant for $T_\Theta(\xi)$ for $\xi \in E$. Fix an $h \in \mathcal{M}$ and $\xi \in E$. Since h is in the range of Θ_1 and Θ_1 intertwines $T_\xi \otimes I_{H_0}$ and $T_\xi \otimes I_{\mathcal{D}_*}$, $(T_\xi \otimes I_{\mathcal{D}_*})h$ is also in the range of Θ_1. Thus $T_\Theta(\xi)h = P_{H(\Theta)}(T_\xi \otimes I_{\mathcal{D}_*})h$ lies in $\mathcal{M}$. Hence $\mathcal{M}$ is an invariant subspace of $H(\Theta)$. Note also that if we start with an equivalent factorization $\Theta = \Theta_1'\Theta_2'$ we get the same subspace $\mathcal{M}$.

It is clear from the decomposition (38) that if we start with an invariant subspace $\mathcal{M}$ and find the factorization $\Theta = \Theta_1\Theta_2$ as above, then the invariant subspace associated to this factorization is the space $\mathcal{M}$ we started with.

Now start with a factorization $\Theta = \Theta_1\Theta_2$ and associate with it the subspace $\mathcal{M} = \Theta_1(\mathcal{F}(E) \otimes_\rho H_0) \ominus \Theta(\mathcal{F}(E) \otimes_{\tau_1} \mathcal{D})$ as above. To this subspace we apply the argument at the beginning of the proof to get a factorization $\Theta = \Theta_1'\Theta_2'$. To do this, we write $\mathcal{N} = \mathcal{M} \oplus \Theta(\mathcal{F}(E) \otimes_{\tau_1} \mathcal{D})$ $(= \Theta_1(\mathcal{F}(E) \otimes_\rho H_0))$ and find a representation ρ' on H_0' and a unitary operator $\Theta_1' : \mathcal{F}(E) \otimes_{\rho'} H_0' \to \mathcal{N}$ that implements a unitary equivalence of the induced representation on $\mathcal{F}(E) \otimes_{\rho'} H_0'$ and the restriction to $\mathcal{N}$ of the induced representation on $\mathcal{F}(E) \otimes_{\tau_2} \mathcal{D}_*$. Setting $V = \Theta_1^*\Theta_1'$ we get a unitary operator from $\mathcal{F}(E) \otimes_{\rho'} H_0'$ onto $\mathcal{F}(E) \otimes_\rho H_0$ that intertwines the induced representations. It is easy to see that such a unitary operator is of the form $I_{\mathcal{F}(E)} \otimes V_0$ for some unitary operator V_0 from H_0' onto H_0 (roughly, V_0 is the restriction of V to H_0' viewed as the wandering subspace of $\mathcal{F}(E) \otimes H_0'$). We thus have $\Theta_1(I_{\mathcal{F}(E)} \otimes V_0) = \Theta_1'$. $\square$

6 An Example: Analytic crossed products

In this section we illustrate some of the results of the previous sections as applied to the special case of correspondences induced from endomorphisms. We shall fix an endomorphism α of a W^*-algebra M and we shall let E be the W^*-correspondence ${}_\alpha M$. That is, as a (right) W^*-module over M, E is M with the inner product defined by the formula $\langle \xi_1, \xi_2 \rangle = \xi_1^*\xi_2$, ξ_1, $\xi_2 \in E$, but the left action is given by α, i.e., $a \cdot \xi$ $(= \varphi(a)\xi) := \alpha(a)\xi$, for $\xi \in E$ and $a \in M$.

The associated Hardy algebra, $H^\infty(E)$, has a particularly attractive description, which we shall develop. Note that for each $k \geq 1$, the correspondence $E^{\otimes k}$ can be identified with ${}_{\alpha^k}M$. The map implementing the isomorphism takes $\xi_1 \otimes \cdots \otimes \xi_k$ to $\alpha^{k-1}(\xi_1)\alpha^{k-2}(\xi_2)\cdots\xi_k$. Thus $\mathcal{F}(E)$ can be identified with the direct sum $\sum_{k=0}^\infty \oplus\, {}_{\alpha^k}M$ (where α^0 is the identity map,

and the zeroth summand, ${}_{\alpha^0}M$, is simply M, viewed as the identity correspondence from M to M).

The action of M on $\mathcal{F}(E)$ given in this form, φ_∞, now written α_∞, is familiar from the theory of crossed products: for $a \in M$, $\alpha_\infty(a)(\xi_k) = (\alpha^k(a)\xi_k)$ for $(\xi_k) \in \mathcal{F}(E)$. On the other hand for $\xi \in E$, the creation operator is given by the formula $T_\xi(\xi_k) = (\theta_k)$ where $\theta_k = \alpha^{k-1}(\xi)\xi_{k-1}$. Note that since

$$T_{a\xi b} = \alpha_\infty(a) T_\xi \alpha_\infty(b),$$

a, $b \in M$ and $\xi \in E$, the operators T_ξ are completely determined by T_1, where 1 is the identity element of M viewed as a vector in E. Evidently, T_1 is a power partial isometry, and assuming that α is unital, which we shall, T_1 is an isometry. We shall write w for T_1. Then $H^\infty(E)$ is simply the σ-weakly closed subalgebra of the W^*-algebra $\mathcal{L}(\mathcal{F}(E))$ generated by $\alpha_\infty(M)$ and w. For historical reasons we shall call this Hardy algebra *the analytic crossed product* determined by M and α and denote it by $M \rtimes_\alpha \mathbb{Z}_+$.

Non-self-adjoint algebras of this form (and closely related algebras) have a long history going back to work of Kadison and Singer [16] and Arveson [2, 3]. In these papers and in most of the subsequent literature, α is assumed to be an *automorphism* of M. However, in [30], Peters studied a related structure associated to an endomorphism of a commutative C^*-algebra and proposed the name *semi-crossed products* for these. They turn out to be examples of tensor algebras and are discussed from this point of view in [22]. The term, *non-self-adjoint crossed product* was introduced in [19], but was changed to *analytic crossed product* some years later in [21] to reflect better their function theoretic aspects. Since we are trying to promote the view that all Hardy algebras as *bona fide* spaces of analytic functions, we shall adopt the term "analytic crossed product" to describe algebras of the form $M \rtimes_\alpha \mathbb{Z}_+$.

Fix a (not-necessarily faithful) representation σ of M on the Hilbert space H. Since $E^{\otimes n}$ may be identified with ${}_{\alpha^n}M$ for all $n \geq 0$, the spaces $E^{\otimes n} \otimes_\sigma H$ may each be identified with H via the Hilbert space isomorphism W_k defined by the formulae

$$W_k(\xi_1 \otimes \cdots \otimes \xi_k \otimes h) = \left\{ \begin{array}{cc} \sigma(\alpha^{k-1}(\xi_1)\alpha^{k-2}(\xi_2)\cdots\xi_k)h, & k > 0 \\ \sigma(\xi_0)h & k = 0 \end{array} \right. , \tag{39}$$

$\xi_i \in E$, $h \in H$. Then the direct sum $W := \sum_{k\geq 0} \oplus W_k$ is a Hilbert space isomorphism from $\mathcal{F}(E) \otimes_\sigma H$ onto $\ell^2(\mathbb{Z}_+, H)$, where $\ell^2(\mathbb{Z}_+, H) := \{\xi : \mathbb{Z}_+ \to$

$H \mid \sum_{k\geq 0} \|\xi(k)\|^2 < \infty\}$. (It will be convenient below to indicate the dependence of W and the W_k on σ by writing W^σ and W_k^σ, but we omit this until necessary.) Define a covariant representation of E on $\ell^2(\mathbb{Z}_+, H)$, denoted (S_H, ψ_H), by the equations

$$(S_H(\xi)x)(k) = \sigma(\alpha^{k-1}(\xi))x(k-1),\ \xi \in E = {}_\alpha M,\ x \in \ell^2(\mathbb{Z}_+, H)$$

and

$$(\psi_H(a)x)(k) = \sigma(\alpha^k(a))x(k),\ a \in M,\ x \in \ell^2(\mathbb{Z}_+, H).$$

Thus, $S_H(1)$ is the unilateral shift (of appropriate multiplicity). Then a moment's reflection using the definition of the representation induced by σ, Definition 2.6, and equations (2) and (3), reveals that W implements a unitary equivalence between the representation of $(M, {}_\alpha M)$ induced by σ and (S_H, ψ_H). That is

$$W\sigma^{\mathcal{F}({}_\alpha M)}(w)W^* = S_H(1)$$

and

$$W\sigma^{\mathcal{F}({}_\alpha M)}(\alpha_\infty(a))W^* = \psi_H(a),$$

$a \in M$.

Consider next an operator $R \in B(\ell^2(\mathbb{Z}_+, H))$ that commutes with the representation $S_H \times \psi_H(M \rtimes_\alpha \mathbb{Z}_+)$. Then since R commutes with the shift $S_H(1)$, it is well known and easy to verify that R must be a block analytic Toeplitz operator. That is, the matrix of R with the direct sum decomposition of $\ell^2(\mathbb{Z}_+, H)$ has this form:

$$R = \begin{pmatrix} R_0 & R_1 & R_2 & & \cdots \\ 0 & R_0 & R_1 & R_2 & \\ 0 & 0 & R_0 & R_1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \\ \vdots & & & & \end{pmatrix}, \tag{40}$$

where each $R_k \in B(H)$. On the other hand, since R commutes with $\psi_H(M)$, a straightforward calculation reveals that each R_k satisfies the equation

$$\sigma(a)R_k = R_k\sigma(\alpha^k(a)), \tag{41}$$

for all $a \in M$, i.e., R_k intertwines σ and $\sigma \circ \varphi^k$. And conversely, every bounded operator R on $\ell^2(\mathbb{Z}_+, H)$ whose matrix with respect to the direct

sum decomposition of $\ell^2(\mathbb{Z}_+, H)$ is a block Toeplitz matrix, as in equation (40), whose entries satisfy equation (41), must commute with the image of $S_H \times \psi_H$.

Suppose now that σ is faithful, so we may form the σ-dual of $E = {}_\alpha M$ and note that $(E^\sigma)^{\otimes k}$ is the σ-dual correspondence of $E^{\otimes k} = {}_{\alpha^k} M$. Hence

$$(E^\sigma)^{\otimes k} = \{\eta : H \to {}_{\alpha^k} M \otimes H \mid \eta\sigma(a) = (\alpha^k(a) \otimes I)\eta, a \in M\}.$$

It follows from the definition of the maps W_k in equation (39) that

$$W_k \cdot (E^\sigma)^{\otimes k} := \{W_k \eta \mid \eta \in (E^\sigma)^{\otimes k}\} = \{z \in B(H) \mid z\sigma(a) = \sigma(\alpha^k(a))z,\ a \in M\}$$

Thus we have substantially proved the following proposition. We leave the remaining details to the reader.

Proposition 6.1 *Suppose $E = {}_\alpha M$, for an endomorphism α of M, and that σ is a faithful representation of M on the Hilbert space H. If $W = \sum_{k \geq 0} \oplus W_k$ is the Hilbert space isomorphism from $\mathcal{F}(E) \otimes_\sigma H$ to $\ell^2(\mathbb{Z}_+, H)$, where the W_k are defined in equation (39) and if $U : \mathcal{F}(E) \otimes_\sigma H \to \mathcal{F}(E^\sigma) \otimes_\iota H$ is the Fourier transform determined by σ, then for all $\eta \in (E^\sigma)^{\otimes k}$, $\sigma(a)W_k\eta = \sigma(\alpha^k(a))W_k\eta$, for all $a \in M$, and*

$$WU^*(T_\eta \otimes I_H)UW^* = \begin{pmatrix} 0 & \cdots & W_k\eta & 0 & \cdots & \\ 0 & 0 & \cdots & W_k\eta & 0 & \cdots \\ & 0 & 0 & \ddots & W_k\eta & \ddots \\ & & 0 & \ddots & & \ddots \\ & & & \ddots & & \\ & & & & & \end{pmatrix}.$$

Further, $WU^*(H^\infty(E^\sigma) \otimes I_H)UW^* = \{R \in B(\ell^2(\mathbb{Z}_+, H)) \mid R$ satisfies equations (40) and (41)$\}$, which is the commutant of $S_H \times \psi_H(H^\infty(E))$.

Suppose now that π is a completely contractive σ-weakly continuous representation of any Hardy algebra, $H^\infty(E)$, on a Hilbert space H, then the associated covariant representation (T, σ) of E is given by the formulae $\sigma = \pi \circ \varphi_\infty$ and $T(\xi) = \pi(T_\xi)$, $\xi \in E$. Consequently, in the present setting, where the Hardy algebra is $M \rtimes_\alpha \mathbb{Z}_+$, if π is a completely contractive σ-weakly continuous representation of $M \rtimes_\alpha \mathbb{Z}_+$ on the Hilbert space H, the

covariant representation (T, σ) of ${}_\alpha M$ on H is determined entirely by σ and the contraction operator $t := T(1) = \pi(w)$. If we let $W_k : E^{\otimes k} \otimes_\sigma H \to H$ be the Hilbert space isomorphism from equation (39) and compute, we find that

$$\begin{aligned}\tilde{T}_k W_k^* \sigma(\alpha^{k-1}(\xi_1)\alpha^{k-2}(\xi_2)\cdots\xi_k))h &= \tilde{T}_k(\xi_1 \otimes \xi_2 \cdots \otimes \xi_k \otimes h) \\ = T(\xi_1)T(\xi_2)\cdots T(\xi_k)h &= T(1)\sigma(\xi_1)T(1)\sigma(\xi_2)\cdots T(1)\sigma(\xi_k)h \\ &= t^k \sigma(\alpha^{k-1}(\xi_1)\alpha^{k-2}(\xi_2)\cdots\xi_k)h.\end{aligned}$$

(In the last equality we used the fact that $t = T(1)$ and the covariance property of the representation). Thus the generalized powers of $\tilde{T}$ are related to the ordinary powers of t through the equation $\tilde{T}_k W_k^* = t^k$ for $k \geq 1$. In particular, we see that $\|\tilde{T}_k^* h\| = \|t^{k*}h\|$ for all $h \in H$. It follows that (T, σ) is a $C_{\cdot 0}$-representation or a c.n.c. representation if and only if t is a $C_{\cdot 0}$-operator or a completely non-coisometric operator.

Also, the defect operators of (T, σ) are related to the defect operators of t via the formulae $(I_H - \tilde{T}\tilde{T}^*)^{1/2} = (I_H - tt^*)^{1/2}$ and $W_1(I_{E\otimes_\sigma H} - \tilde{T}^*\tilde{T})^{1/2}W_1^* = (I_H - t^*t)^{1/2}$. Hence, if we form $\tau_1 := \sigma \circ \alpha|\mathcal{D}$ where, as usual, $\mathcal{D} = \overline{(I_{E\otimes_\sigma H} - \tilde{T}^*\tilde{T})^{1/2}(E \otimes_\sigma H)}$, and if we form $W^{\tau_1} : \mathcal{F}(E) \otimes_{\tau_1} \mathcal{D} \to \ell^2(\mathbb{Z}_+, \mathcal{D})$ and follow it with $I \otimes W_1$ mapping $\ell^2(\mathbb{Z}_+, \mathcal{D})$ onto $\ell^2(\mathbb{Z}_+, \mathcal{D}_t)$, where $\mathcal{D}_t = \overline{(I_H - t^*t)^{1/2}H}$ is the defect space of t, then $\mathcal{W}_v := I_H \oplus (I \otimes W_1)W^{\tau_1}$ is a Hilbert space isomorphism mapping the Hilbert space of the minimal isometric dilation (V, ρ) of (T, σ) onto the Hilbert space of the minimal isometric dilation of t, vis., $H \oplus \ell^2(\mathbb{Z}_+, \mathcal{D}_t)$. Further, we have $\mathcal{W}_v V(1)\mathcal{W}_v^* = v$, where

$$v = \begin{pmatrix} t & 0 & 0 & \cdots \\ d & 0 & 0 & \cdots \\ 0 & I_{\mathcal{D}_t} & 0 & \\ 0 & 0 & I_{\mathcal{D}_t} & \\ & & & \ddots \\ & & & \end{pmatrix},$$

and $d := (I_H - t^*t)^{1/2}$.

Now consider the characteristic operator of (T, σ), $(\Theta_T, \mathcal{D}, \mathcal{D}_*, \tau_1, \tau_2)$ and identify (T, σ) with its canonical model using Theorem 3.25. Recall from Remark 3.11 that our notation remains consistent; this new τ_1 is still the restriction of $\sigma \circ \alpha$ to $\mathcal{D}$; τ_2 is the restriction of σ to $\mathcal{D}_*$. Even though the

defect space $\mathcal{D}_*$ for (T,σ) is the same as the defect space $\mathcal{D}_{*t} := \overline{(I-tt^*)^{1/2}H}$, we shall continue to distinguish notationally between them. Thus $t = T(1)$ is the operator which, in the notation of Theorem 3.25, would be denoted $T_{\hat{\Theta}_T}(1)$ and similarly the minimal isometric dilation (V,ρ) of (T,σ) would be denoted $(V_{\hat{\Theta}_T}, \rho_{\hat{\Theta}_T})$, etc. However, this notation is ponderous and so we shall drop the subscript $\hat{\Theta}_T$. We shall write $\mathcal{W}_*$ for $\mathcal{W}^{\tau_2}$, so that $\mathcal{W}_*$ is a Hilbert space isomorphism from $\mathcal{F}(E)\otimes_{\tau_2}\mathcal{D}_*$ onto $\ell^2(\mathbb{Z}_+,\mathcal{D}_{*t})$ such that

$$\mathcal{W}_*(w\otimes I_{\mathcal{D}_*}) = S_{\mathcal{D}_{*t}}\mathcal{W}_*$$

where $S_{\mathcal{D}_{*t}}$ is the unilateral shift on $\ell^2(\mathbb{Z}_+,\mathcal{D}_{*t})$. We also write $\mathcal{W}_1$ for $(I\otimes W_1^{\tau_1})W^{\tau_1}$, which is a Hilbert space isomorphism from $\mathcal{F}(E)\otimes_{\tau_1}\mathcal{D}$ onto $\ell^2(\mathbb{Z}_+,\mathcal{D}_t)$ that satisfies the equation

$$\mathcal{W}_1(w\otimes I_{\mathcal{D}}) = S_{\mathcal{D}_t}\mathcal{W}_1,$$

where for $S_{\mathcal{D}_t}$ is the unilateral shift on $\ell^2(\mathbb{Z}_+,\mathcal{D}_t)$. The characteristic operator Θ_T maps $\mathcal{F}(E)\otimes_{\tau_1}\mathcal{D}$ to $\mathcal{F}(E)\otimes_{\tau_2}\mathcal{D}_*$ and intertwines the induced representations, $\tau_1^{\mathcal{F}(E)}$ and $\tau_2^{\mathcal{F}(E)}$. Thus, if we set $\Theta := \mathcal{W}_*\Theta_T\mathcal{W}_1^{-1}$, we obtain a contraction from $\ell^2(\mathbb{Z}_+,\mathcal{D}_t)$ to $\ell^2(\mathbb{Z}_+,\mathcal{D}_{*t})$ that intertwines $S_{\mathcal{D}_{*t}}$ and $S_{\mathcal{D}_t}$. We shall write Δ_T for $(I-\Theta_T^*\Theta_T)^{1/2}$ and Δ for $(I-\Theta^*\Theta)^{1/2}$, so $\mathcal{W}_1\Delta_T\mathcal{W}_1^{-1} = \Delta$. Also, we shall write $\mathcal{W}_\Delta$ for the restriction of $\mathcal{W}_1$ to $\overline{\Delta_T(\mathcal{F}(E)\otimes_{\tau_1}\mathcal{D})}$, obtaining a Hilbert space isomorphism from this space onto $\overline{\Delta\ell^2(\mathbb{Z}_+,\mathcal{D}_t)}$. Consequently, $\mathcal{W}$ which we shall define to be $\mathcal{W}_*\oplus\mathcal{W}_\Delta$ is a Hilbert space isomorphism from $K(\Theta_T)$, which recall from Theorem 3.25 is $(\mathcal{F}(E)\otimes_{\tau_2}\mathcal{D}_*)\oplus\overline{\Delta_T(\mathcal{F}(E)\otimes_{\tau_1}\mathcal{D})}$, onto $\ell^2(\mathbb{Z}_+,\mathcal{D}_{*t})\oplus\overline{\Delta\ell^2(\mathbb{Z}_+,\mathcal{D}_t)}$.

Recall next the definition of $S_{\hat{\Theta}_T}(\cdot) := S$, from Lemma 3.23, and write S for the isometry $S(1)$. (Actually, S is unitary as we shall see in a moment.) Then if $\tilde{S}$ is defined on $\overline{\Delta\ell^2(\mathbb{Z}_+,\mathcal{D}_t)}$ by the formula $\tilde{S}(\Delta\xi) = \Delta S_{\mathcal{D}_t}\xi$, then, as an easy calculation shows, $\mathcal{W}_\Delta$ implements a unitary equivalence between S and $\tilde{S}$. Consequently, $\mathcal{W}$ implements a unitary equivalence between $S_{\mathcal{D}_{*t}}\oplus\tilde{S}$ acting on $\ell^2(\mathbb{Z}_+,\mathcal{D}_{*t})\oplus\overline{\Delta\ell^2(\mathbb{Z}_+,\mathcal{D}_t)}$. Thus, it looks like $\mathcal{W}$ implements a unitary equivalence between the minimal isometric dilation $v = V(1)$ for t and the isometry that occurs in the Sz.-Nagy-Foiaş model for t in [41].[1] But

[1]Strictly speaking to identify fully the constructs of the Sz.-Nagy-Foiaş theory, we need to transfer the discussion from ℓ^2-spaces on $\mathbb{Z}$ to L^2-spaces on $\mathbb{T}$ via the Fourier transform. We omit this detail. However, the whole theory has been developed on $\mathbb{Z}$ by Douglas in [11].

$S_{\mathcal{D}_{*t}} \oplus \tilde{S}$ is not quite the Sz.-Nagy-Foiaş model isometry. The point is that the model that Sz.-Nagy and Foiaş produce acts on $\ell^2(\mathbb{Z}_+, \mathcal{D}_{*t}) \oplus \overline{\tilde{\Delta}\ell^2(\mathbb{Z}, \mathcal{D}_t)}$, where $\ell^2(\mathbb{Z}, \mathcal{D}_t)$ consists of all square summable $\mathcal{D}_t$-valued functions on the *integers* $\mathbb{Z}$, $\tilde{\Delta}$ is an operator that we describe in a second and the part of the model that acts on $\overline{\tilde{\Delta}\ell^2(\mathbb{Z}, \mathcal{D}_t)}$ is the (restriction of the) bilateral shift. The difference lies in the definition of $\tilde{\Delta}$. Note that since Θ intertwines $S_{\mathcal{D}_{*t}}$ and $S_{\mathcal{D}_t}$, Θ has a *unique* extension to an operator $\tilde{\Theta}$ from $\ell^2(\mathbb{Z}, \mathcal{D}_t)$ to $\ell^2(\mathbb{Z}, \mathcal{D}_{*t})$ that intertwines the two *bilateral* shifts. We simply let $\tilde{\Delta} = (I - \tilde{\Theta}^*\tilde{\Theta})^{1/2}$. Then the piece Sz.-Nagy and Foiaş build for their model is $\overline{\tilde{\Delta}\ell^2(\mathbb{Z}, \mathcal{D}_t)}$. However, in terms of $\tilde{\Theta}$, $\Delta = (I - P\tilde{\Theta}^*P\tilde{\Theta})^{1/2}|_{\ell^2(\mathbb{Z}_+, \mathcal{D}_t)}$, so on the face of it, one would expect $\overline{\Delta\ell^2(\mathbb{Z}_+, \mathcal{D}_t)}$ and $\overline{\tilde{\Delta}\ell^2(\mathbb{Z}, \mathcal{D}_t)}$ to be different. Neverthless, if we assume that our representation (T, σ) is c.n.c., as we shall, then the map that takes $\ell^2(\mathbb{Z}_+, \mathcal{D}_t)$ to $\overline{\tilde{\Delta}\ell^2(\mathbb{Z}, \mathcal{D}_t)}$ by sending a vector of the form $\Delta\xi$ to $\tilde{\Delta}\tilde{\xi}$, where $\tilde{\xi}$ is the extension of ξ to all of $\mathbb{Z}$, which is zero on the negative integers, is in fact a Hilbert space isomorphism that intertwines $\tilde{S}$ on $\ell^2(\mathbb{Z}_+, \mathcal{D}_t)$ and the restriction of the bilateral shift to $\overline{\tilde{\Delta}\ell^2(\mathbb{Z}, \mathcal{D}_t)}$. This is the content, really, of part (ii) of Lemma 3.31, which gives meaning to the term "predictable". Thus, if we incorporate this additional Hilbert space isomorphism $(\Delta\xi \mapsto \tilde{\Delta}\tilde{\xi})$ into the definition of $\mathcal{W}$, then we have proved most of the following theorem. The remaining details are easy to supply and so will be omitted.

Theorem 6.2 *Let π be a completely contractive, σ-weakly continuous representation of the analytic crossed product $M \rtimes_\alpha \mathbb{Z}_+$ on a Hilbert space H such that $t = \pi(w) = T(1)$ is a c.n.c. contraction, where (T, σ) is the associated covariant representation, and let $(\Theta_T, \mathcal{D}, \mathcal{D}_*, \tau_1, \tau_2)$ be the characteristic operator attached to this representation. Then the Hilbert space isomorphism $\mathcal{W}$ just described, viewed as a map from the space $K(\hat{\Theta}_T)$ of the minimal isometric dilation of (T, σ) to the shift space $\ell^2(\mathbb{Z}_+, \mathcal{D}_{*t}) \oplus \overline{\tilde{\Delta}\ell^2(\mathbb{Z}, \mathcal{D}_t)}$ maps all parts of the model space for (T, σ) to the corresponding parts of Sz.-Nagy-Foiaş model space for t, i.e., the operator $\Theta = \mathcal{W}_*\Theta_T\mathcal{W}^{-1}$ described above is equivalent to the characteristic operator function of the operator t described in [41].*

Concluding Remarks 6.3

(i) In view of Theorem 6.2, it appears that for analytic crossed products, at least, one may extend the model developed in Theorem 3.25 to get

a unitary *dilation for a c.n.c. representation* (T, σ) *of the algebra. That is, thinking of the isometric dilation* (V, ρ) *for* (T, σ) *as acting on* $\ell^2(\mathbb{Z}_+, \mathcal{D}_{*t}) \oplus \overline{\tilde{\Delta}\ell^2(\mathbb{Z}, \mathcal{D}_t)}$, $v := V(1)$ *is an isometry that satisfies the equation* $v\rho \circ \alpha(a) = \rho(a)v$ *for all* $a \in M$. *The minimal unitary* extension *of* v *is the (restriction of the) bilateral shift acting* $\ell^2(\mathbb{Z}, \mathcal{D}_{*t}) \oplus \overline{\tilde{\Delta}\ell^2(\mathbb{Z}, \mathcal{D}_t)}$. *However, while* v *extends to a unitary* w, *say, on* $\ell^2(\mathbb{Z}, \mathcal{D}_{*t}) \oplus \overline{\tilde{\Delta}\ell^2(\mathbb{Z}, \mathcal{D}_t)}$, *it may not be possible to extend* ρ *to a representation* $\tilde{\rho}$ *on this space so that the equation* $w\tilde{\rho} \circ \alpha(a) = \tilde{\rho}(a)w$ *also holds for all* $a \in M$. *If such a* $\tilde{\rho}$ *were to exist, then it would have a natural extension to the* C^**-inductive limit of the system built from* M *and the powers of* α *as described in [40]. Simple examples show that this need not be the case. We intend to take this matter up in a future study.*

(ii) *The example studied in this section may seem very special. However, thanks to our investigation in [25], we may assert that under technical conditions that we ignore here, every* W^**-correspondence over a von Neumann algebra is Morita equivalent to one that comes from an endomorphism of another, possibly different, von Neumann algebra. Thus, up to Morita equivalence, all Hardy algebras are analytic crossed products. We intend take this matter up also in a future study.*

(iii) *As we noted in Theorem 6.2, the characteristic function* $\hat{\Theta}_T$ *of the representation* (T, σ) *is equivalent to the characteristic operator function* Θ *of* $t = T(1)$ *(after one takes the Fourier transform that identifies* ℓ^2 *with* $L^2(\mathbb{T})$ *and identifies* Θ *as a function, rather than as an operator.). Classically,* Θ *is an analytic function from the open unit disc* $\mathbb{D}$ *in* $\mathbb{C}$ *to* $B(\mathcal{D}_t, \mathcal{D}_{*t})$. *On the other hand, because* $\hat{\Theta}_T$ *is an element of* $H^\infty(E^\tau)$, *where* $(\mathcal{G}, \tau)$ *is the supplement of* τ_1 *and* τ_2 *that we fixed in the discussion just before equation (15),* $\hat{\Theta}_T$ *has a Taylor or Fourier expansion*

$$\hat{\Theta}_T \sim T_{\eta_0} + T_{\eta_1} + \cdots,$$

where the $\eta_i \in (E^\tau)^{\otimes i}$. *As we show in [28] using the gauge group, the arithmetic means of this series converge weak-$*$ to* $\hat{\Theta}_T$. *As we noted above,* $W_i \cdot (E^\sigma)^{\otimes i} := \{W_i\eta \mid \eta \in (E^\sigma)^{\otimes i}\} = \{z \in B(\mathcal{G}) \mid z\sigma(a) = \sigma(\alpha^i(a))z,\ a \in M\}$. *To compute the* $W_i\eta_i \in B(\mathcal{G})$, *we may appeal to the analysis leading to Theorem 3.21 or to the result of the calculation there*

to conclude that $W_0\eta_0 = -t|\mathcal{D}$, $W_1\eta_1 = \Delta_*\Delta|\mathcal{D}$, $W_2\eta_2 = \Delta_* t^*\Delta|\mathcal{D}$, $\cdots$. *So, if we evaluate* Θ_T *on the open unit ball of* $E = {}_\alpha M$ *using the formula from Theorem 3.21, then a straightforward calculation based on the analysis we have made and the definition of the characteristic operator function for* t *from [41] shows that if* ξ_0 *denotes the identity operator in* M*, but viewed as a vector in* E*, then for all complex numbers* z*,* $|z| < 1$*,*

$$\Theta(\overline{z}) = \hat{\Theta}_T(z\xi_0).$$

(The reason for $\overline{z}$ *and not* z *is an artifact of the role that elements in the dual play in the representations of the algebras and need not concern us here.) Thus,* $\hat{\Theta}_T$ *is effectively determined on the one dimensional slice* $\{z\xi_0 \mid |z| < 1\}$*. Of course, this is fairly evident from Theorem 3.21 and the fact that* ξ_0 *is a cyclic vector for* E *as a right module over* M*.*

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